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ON THE OSCILLATION OF A THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH NEUTRAL TYPE

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Abstract: In this article, we investigate the oscillation behavior of the solutions of the third-order nonlinear differential equation with neural type of the form

$$\Big(a_1(t)\big(a_2(t)Z'(t)\big)'\Big)' + q(t)f\big(x(\sigma(t))\big) = 0, \quad t \geq t_0 > 0,$$

where $Z(t) := x(t) + p(t)x^{\alpha}(\tau(t))$. Some new oscillation results are presented that extend those results given in the literature.
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1. Introduction

Consider the third order non-linear neutral delay differential equation

$$\Big(a_1(t)\big(a_2(t)Z'(t)\big)'\Big)' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 > 0, \tag{E}$$

where $Z(t) := x(t) + p(t)x^{\alpha}(\tau(t))$ and $0 < \alpha \leq 1$ is a ratio of odd positive integers. Throughout this paper, without further mention, let

(A_1) $a_i(t) \in C([t_0, +\infty))$, $a_i(t) > 0$ for $i = 1, 2$ and $p(t), q(t) \in C([t_0, +\infty))$, $q(t) > 0$;

(A_2) $\tau(t) \in C([t_0, +\infty))$, $\tau(t) \leq t$, $\sigma(t) \in C([t_0, +\infty))$, $\sigma(t) \leq t$;

(A_3) f is nondecreasing and $uf(u) \geq k > 0$ for $u \neq 0$ and $\lim\limits_{t\to+\infty} \tau(t) = \lim\limits_{t\to+\infty} \sigma(t) = \infty$.

By a solution of equation (E) we mean a nontrivial real valued function $x(t) \in C([T_x, \infty)), T_x \geq t_0$, which has the property $Z'(t) \in C^1([T_x, \infty))$, $a_2(t)Z'(t) \in C^1([T_x, \infty))$, $a_1(t)(a_2(t)Z'(t))' \in C^1([T_x, \infty))$ and satisfies (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise, it is said to be non-oscillatory. Equation (E) is called almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically.

In the last years, a great deal of interest in oscillatory properties of neutral functional differential equations has been shown, we refer the reader to [1–8] and the references cited therein. A number was established. By (10.4) we obtain that

$$\big((\tau - \mathrm{link})_0[E], \mathbb{T}_0(E|\,\tau)\big) = \big((\tau - \mathrm{link})_0[E], \mathbb{T}_*(E|\,\tau)\big) \tag{10.5}$$

is a zero-dimensional supercompactum. In addition, by Proposition 4

$$\mathbb{F}_0^*(\tau) \in \mathbf{C}_{(\tau-\mathrm{link})_0[E]}[\mathbb{T}_0(E|\,\tau)];$$

so, $\mathbb{F}_0^*(\tau)$ is the closed in the supercompactum (10.5). We obtain that compactum (10.3) is a closed subspace of the supercompactum (10.5).

11. Conclusion

We reviewed two BTS. In the first case point of BTS are MLS and, in the second case, similar points are u/f of a set lattice. It is established that the second BTS can be considered as a subspace of the first BTS. We indicated the natural variants of our lattice for which the above-mentioned BTS are degenerate and, opposite, the variants with degeneracy of the corresponding BTS is absent. Our consideration is connected with ideas of supercompactness and superextension of a TS. For degenerate BTS the corresponding space of MLS is a supercompactum. Under consideration of the lattice of closed MLS, we obtain a non-degenerate BTS typically.

REFERENCES

1. **De Groot J.** Superextensions and supercompactness // Proc. I. Intern. Symp. on extension theory of topological structures and its applications. Berlin: VEB Deutscher Verlag Wis., 1969. P. 89–90.
2. **Van Mill J.** Supercompactness and Wallman spaces. Amsterdam. Math. Center Tract., 85. 1977.
3. **Strok M. and Szymanski.** Compact metric spaces have binary bases // Fund. Math., 1975. Vol. 89. P. 81–91.
4. **Fedorchuk V.V., Filippov V.V.** Obshhaya topologiya. Osnovnyie konstrukzii. M.: Fismatlit, 2006. [in Russian]
5. **Chentsov A.G.** Ultrafilters and maximal linked systems of sets // Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 2017. Vol. 27, no. 3. P. 365–388. [in Russian]
6. **Chentsov A.G.** Filters and ultrafilters in the constructions of attraction sets // Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 2011. No. 1, P. 113–142. [in Russian]
7. **Chentsov A.G.** Tier mappings and ultrafilter-based transformations // Trudy Inst. Mat. i Mekh. UrO RAN, 2012. Vol. 18, no. 4. P. 298–314. [in Russian]
8. **Chentsov A.G.** Compactifiers in extension constructions for reachability problems with constraints of asymptotic nature // Steklov Inst. Math. 2017. Vol. 296, suppl. 1. P. 102–118. DOI: 10.1134/S0081543817020109
9. **Dvalishvili B.P.** Bitopological Spaces: Theory, Relations with Generalized Algebraic Structures, and Applications. Nort-Holland. Mathematics studies. 2005.
10. **Kuratowski K., Mostowski A.** Set theory. Amserdam: North-Holland, 1967.
11. **Alexanfroff P.S.** Vvedenie v teoriyu mnogestv i obshhuju topologiyu. M.: Editorial URSS, 2004. [in Russian]
12. **Alexandroff A.D.** Additive set-functions in abstract spaces // Mathematics of the USSR-Sbornik. 1940. Vol. 8, no. 2. P. 307–348.
13. **Engelking R.** General topology. Warsaw: PWN. 1977.
14. **Chentsov A.G.** Attraction sets in abstract attainability problems: equivalent representations and basic properties // Russ Math., 2013. Vol. 57, no. 28. DOI:10.3103/S1066369X13110030
15. **Chentsov A.G., Pytkeev E.G.** Some topological structures of extensions of abstract reachability problems // Proc. Steklov Inst. Math., 2016. 292, suppl. 1. P. 36–54. DOI: 10.1134/S0081543816020048
16. **Chentsov A.G.** Superextension as bitopological space // Izv. IMI UdGU, 2017. Vol. 49. P. 55–79. [in Russian]
17. **Chentsov A.G.** To the validity of constraints in the class of generalized elements // Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 2014. Vol. 3. P. 90–109. [in Russian]

of course, (9.3) is a nonempty zero-dimensional compactum. Moreover, by (9.1)

$$\mathcal{L} = \mathbf{C}_E[\mathcal{L}]. \tag{9.4}$$

Therefore, the sets $(\mathcal{L} - \mathrm{link})^0_{\mathrm{op}}[E|\, L],\ L \in \mathcal{L}$, are defined. In addition,

$$(\mathcal{L} - \mathrm{link})^0_{\mathrm{op}}[E|\, L] = (\mathcal{L} - \mathrm{link})^0[E|\, L]$$

under $L \in \mathcal{L}$. We recall that, under $\Lambda \in \mathcal{L}$, the inclusion $E \setminus \Lambda \in \mathcal{L}$ is realized; and what is more, by (3.13)

$$(\mathcal{L} - \mathrm{link})^0[E|\, E \setminus \Lambda] = (\mathcal{L} - \mathrm{link})_0[E] \setminus (\mathcal{L} - \mathrm{link})^0[E|\, \Lambda].$$

As a simple corollary, in our case, the equality

$$\mathfrak{C}^*_0[E; \mathcal{L}] = \mathfrak{C}^0_{\mathrm{op}}[E; \mathcal{L}]$$

is realized. Therefore, by (3.15) and (4.2)

$$\mathbb{T}_0(E|\, \mathcal{L}) = \mathbb{T}_*(E|\, \mathcal{L}). \tag{9.5}$$

So, by (9.5) we have the following important property: TS

$$\big((\mathcal{L} - \mathrm{link})_0[E], \mathbb{T}_0(E|\, \mathcal{L})\big) = \big((\mathcal{L} - \mathrm{link})_0[E], \mathbb{T}_*(E|\, \mathcal{L})\big) \tag{9.6}$$

is a nonempty supercompactum. In particular, (9.6) is a nonempty compactum.

Proposition 9. *The set $\mathbb{F}^*_0(\mathcal{L})$ is closed in TS* (9.6)*:*

$$\mathbb{F}^*_0(\mathcal{L}) \in \mathbf{C}_{(\mathcal{L} - \mathrm{link})_0[E]}[\mathbb{T}_0(E|\, \mathcal{L})]. \tag{9.7}$$

The corresponding proof follows from Proposition 4 (indeed, for (9.6) we have the separability property). In connection with Proposition 9, we recall (3.31).

10. Open maximal linked systems

In this section, we suppose that

$$\mathcal{L} = \tau, \tag{10.1}$$

where $\tau \in (\mathrm{top})[E]$. So, $(E, \mathcal{L}) = (E, \tau)$ is a TS. We consider the lattice of open sets. In this connection, we recall (see [15, Section 8]) that

$$\mathbf{T}^0_\tau[E] = \mathbf{T}^*_\tau[E]. \tag{10.2}$$

Of course, in the form of

$$\big(\mathbb{F}^*_0(\tau), \mathbf{T}^0_\tau[E]\big) = \big(\mathbb{F}^*_0(\tau), \mathbf{T}^*_\tau[E]\big), \tag{10.3}$$

we obtain a nonempty zero-dimensional compactum of open u/f (using (10.1), we consider u/f consisting of open sets as open u/f). On the other hand, by (10.1) we can consider MLS consisting of open sets. We call such MLS open also (recall that $(\mathrm{top})[E] \subset (\mathrm{LAT})_0[E]$). By [5, Proposition 9.1]

$$(\tau - \mathrm{link})_0[E] \setminus (\tau - \mathrm{link})^0[E|\, G] = (\tau - \mathrm{link})^0[E|\, E \setminus \mathrm{cl}(G, \tau)] \quad \forall\, G \in \tau.$$

With employment of this property, in [5, Proposition 9.2], the equality

$$\mathbb{T}_0(E|\, \tau) = \mathbb{T}_*(E|\, \tau) \tag{10.4}$$

So, by (8.8) the BTS (8.7) is non-degenerate. Moreover, we have topologies

$$\big(\mathbf{T}^0_{\widetilde{\mathcal{L}}\cup\{E\}}[E] \in (\mathrm{top})[\mathbb{F}_0^*(\widetilde{\mathcal{L}} \cup \{E\})]\big)\&\big(\mathbf{T}^*_{\widetilde{\mathcal{L}}\cup\{E\}}[E] \in (\mathrm{top})[\mathbb{F}_0^*(\widetilde{\mathcal{L}} \cup \{E\})]\big).$$

In addition, in the form of the triplet

$$\big(\mathbb{F}_0^*(\widetilde{\mathcal{L}} \cup \{E\}), \mathbf{T}^0_{\widetilde{\mathcal{L}}\cup\{E\}}[E], \mathbf{T}^*_{\widetilde{\mathcal{L}}\cup\{E\}}[E]\big) \tag{8.9}$$

we have BTS. Of course, (8.9) is a variant of BTS (2.12). By (7.8), (8.4), and (8.5) we obtain that

$$\mathbf{T}^0_{\widetilde{\mathcal{L}}\cup\{E\}}[E] \neq \mathbf{T}^*_{\widetilde{\mathcal{L}}\cup\{E\}}[E]. \tag{8.10}$$

So, by (8.10) the BTS (8.9) is non-degenerate. We recall that, in Section 1, the concrete examples of the realization of (8.9) and (8.10) were identified (see Examples 1.1–1.4). Now, we consider yet one example of such type.

Example 8.1. Let $\sqsubseteq$ be a direction on the (nonempty) set E. So, we consider the case of nonempty directed set $(E, \sqsubseteq)$. Suppose that

$$(\sqsubseteq -\mathrm{Ma})_E[Y] \stackrel{\triangle}{=} \{z \in E|\, y \sqsubseteq z \quad \forall\, y \in Y\} \quad \forall\, Y \in \mathcal{P}(E).$$

Then $\mathfrak{M}[E; \sqsubseteq] \stackrel{\triangle}{=} \{Y \in \mathcal{P}(E)|\, (\sqsubseteq -\mathrm{Ma})_E[Y] \neq \emptyset\}$ is the family of all majorized subsets of E. Since $E \neq \emptyset$, we have the obvious property $\emptyset \in \mathfrak{M}[E; \sqsubseteq]$ (moreover, by the choice of $\sqsubseteq$ we obtain that $\{x; y\} \in \mathfrak{M}[E; \sqsubseteq] \quad \forall\, x \in E \quad \forall\, y \in E$). From properties of directed sets, the statement $\mathfrak{M}[E; \sqsubseteq\,] \in (\mathrm{LAT})[E]$ is realized. It is obvious that $\{x\} \in \mathfrak{M}[E; \sqsubseteq] \quad \forall\, x \in E$. Finally,

$$\bigcap_{\mathcal{H}\in\mathfrak{H}} \mathcal{H} \in \mathfrak{M}[E; \sqsubseteq] \quad \forall\, \mathfrak{H} \in \mathcal{P}'(\mathfrak{M}[E; \sqsubseteq]).$$

As a corollary, by (1.17) we obtain the implication

$$(E \notin \mathfrak{M}[E; \sqsubseteq]) \Longrightarrow (\mathfrak{M}[E; \sqsubseteq] \in (\downarrow -\mathrm{LAT})^0[E]).$$

So, under $E \notin \mathfrak{M}[E; \sqsubseteq]$, in the form of $\mathfrak{M}[E; \sqsubseteq]$, we obtain yet one variant of the family of $(\downarrow -\mathrm{LAT})^0[E]$: $\mathfrak{M}[E; \sqsubseteq] \in (\downarrow -\mathrm{LAT})^0[E]$. □

9. Measurable space with algebra of sets

Recall that by (1.6) and (1.7) $(\mathrm{alg})[E] \subset (\mathrm{LAT})_0[E]$. Using this property, in the present section, we consider the case

$$\mathcal{L} \in (\mathrm{alg})[E]. \tag{9.1}$$

By (9.1) we have that (in the present section) $(E, \mathcal{L})$ is a measurable space with algebra of sets. We recall Remark 2.1: in the form of

$$\big(\mathbb{F}_0^*(\mathcal{L}), (\mathbb{UF})[E; \mathcal{L}]\big),$$

a measurable space with algebra of sets is realized also. Moreover, we have BTS (2.12). But, by [6, Proposition 9.2] this BTS is degenerate:

$$\mathbf{T}^0_{\mathcal{L}}[E] = \mathbf{T}^*_{\mathcal{L}}[E]. \tag{9.2}$$

By (9.2) we obtain the following equality of TS:

$$\big(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}^0_{\mathcal{L}}[E]\big) = \big(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}^*_{\mathcal{L}}[E]\big); \tag{9.3}$$

At the same time, we have (see [5, § 7]) the property

$$\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^0_{\mathbf{C}_E[\tau]}[E]\big)=\Phi_{\mathbf{C}_E[\tau]}\big(\mathrm{cl}(A,\tau)\big)\quad\forall A\in\mathcal{P}(E). \tag{7.5}$$

So, by (7.5) the following statement is realized: TS (2.10) «feels» subsets of E accurate to closure. We recall [5, (7.3)]: for $A\in\mathcal{P}(E)$ and $x_*\in\mathrm{cl}(A,\tau)\setminus A$

$$(\mathbf{C}_E[\tau]-\mathrm{triv})[x_*]\in\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^0_{\mathbf{C}_E[\tau]}[E]\big)\setminus\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^*_{\mathbf{C}_E[\tau]}[E]\big). \tag{7.6}$$

With employment of (7.6), we obtain (see [5, (7.4)]) in our case

$$\mathbf{C}_E[\tau]=\big\{A\in\mathcal{P}(E)|\,\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^0_{\mathbf{C}_E[\tau]}[E]\big)=\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^*_{\mathbf{C}_E[\tau]}[E]\big)\big\}. \tag{7.7}$$

Finally, by [5, Theorem 7.1] we obtain the following implication:

$$\big(\tau\neq\mathcal{P}(E)\big)\Longrightarrow(\mathbf{T}^0_{\mathbf{C}_E[\tau]}[E]\neq\mathbf{T}^*_{\mathbf{C}_E[\tau]}[E]). \tag{7.8}$$

So, for (7.1) and nondiscrete T_1-space (E,τ), BTS (2.12) is nondegenerate. From (4.18) and (7.8), we obtain that

$$\big(\tau\neq\mathcal{P}(E)\big)\Longrightarrow\big(\mathbb{T}_0(E|\,\mathbf{C}_E[\tau])\neq\mathbb{T}_*(E|\,\mathbf{C}_E[\tau])\big). \tag{7.9}$$

We use (7.8) and (7.9) in connection with lattices of the family (1.17).

8. Some particular cases

In this section, we fix a lattice

$$\widetilde{\mathcal{L}}\in(\downarrow-\mathrm{LAT})^0[E]. \tag{8.1}$$

Then, by (1.18) we obtain that $\widetilde{\mathcal{L}}\cup\{E\}\in(\mathrm{clos})[E]$ and (in particular) $\widetilde{\mathcal{L}}\cup\{E\}\in(\mathrm{LAT})_0[E]$. In addition,

$$\tau^0_{\widetilde{\mathcal{L}}}[E]=\mathbf{C}_E[\widetilde{\mathcal{L}}\cup\{E\}]=\mathbf{C}_E[\widetilde{\mathcal{L}}]\cup\{\emptyset\}\in(\mathcal{D}-\mathrm{top})[E] \tag{8.2}$$

realizes the following T_1-space:

$$(E,\tau^0_{\widetilde{\mathcal{L}}}[E]). \tag{8.3}$$

We recall that (see Section 1), for (8.2) and (8.3), the following property takes place: (8.3) is not T_2-space. From (8.2), we have the equality

$$\widetilde{\mathcal{L}}\cup\{E\}=\mathbf{C}_E\big[\tau^0_{\widetilde{\mathcal{L}}}[E]\big] \tag{8.4}$$

(see (1.21)). In addition, by (1.22) we obtain that

$$\tau^0_{\widetilde{\mathcal{L}}}[E]\neq\mathcal{P}(E). \tag{8.5}$$

We recall that by (1.20) $\widetilde{\mathcal{L}}\cup\{E\}\in(\mathcal{D}-\mathrm{clos})[E]$. In addition,

$$\Big(\mathbb{T}_0(E|\,\widetilde{\mathcal{L}}\cup\{E\})\in(\mathrm{top})\big[\big((\widetilde{\mathcal{L}}\cup\{E\})-\mathrm{link}\big)_0[E]\big]\Big)\&\Big(\mathbb{T}_*(E|\,\widetilde{\mathcal{L}}\cup\{E\})\in(\mathrm{top})\big[\big((\widetilde{\mathcal{L}}\cup\{E\})-\mathrm{link}\big)_0[E]\big]\Big). \tag{8.6}$$

In the form of the triplet

$$\Big(\big((\widetilde{\mathcal{L}}\cup\{E\})-\mathrm{link}\big)_0[E],\mathbb{T}_0(E|\,\widetilde{\mathcal{L}}\cup\{E\}),\mathbb{T}_*(E|\,\widetilde{\mathcal{L}}\cup\{E\})\Big), \tag{8.7}$$

we obtain a BTS. Of course, (8.7) is a variant of BTS (4.20). By (7.9), (8.4), and (8.5) we obtain that

$$\mathbb{T}_0(E|\,\widetilde{\mathcal{L}}\cup\{E\})\neq\mathbb{T}_*(E|\,\widetilde{\mathcal{L}}\cup\{E\}). \tag{8.8}$$

6. Some additions

In this sections, at first, we consider questions meaningful of a duality for families $\mathfrak{C}_0^*[E;\mathcal{L}]$ and $\mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}]$. For this, we recall that (see Section 3)

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{6.1}$$

As a corollary, by (4.4) and (6.1) we have the property

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_*(E|\,\mathcal{L})\big]\cap(\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{6.2}$$

Proposition 8. *The family* $\mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}]$ *is a closed subbase of the TS* (4.3)*:*

$$\mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_*(E|\,\mathcal{L})\big]. \tag{6.3}$$

P r o o f. We recall (3.24). So, by (4.5) we have the following statement

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_{\emptyset}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_*(E|\,\mathcal{L})\big]:\ \mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}] = \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}\big[\mathfrak{C}_0^*[E;\mathcal{L}]\big].$$

Then, by (1.16) we obtain (6.3). □

From (3.17) and Proposition 8 we have the following property

$$\mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]\cap(\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_*(E|\,\mathcal{L})\big]. \tag{6.4}$$

In (6.2) and (6.4), we obtain a duality of subbases.

7. Bitopological space of closed ultrafilters and maximal linked systems

We recall (5.6). Then, by this property the topologies $\mathbf{T}_{\mathcal{L}}^0[E]$ and $\mathbf{T}_{\mathcal{L}}^*[E]$ are similar (later, we show that in many cases the above-mentioned topologies are equal). But, now we consider the variant of the set lattice for which the above-mentioned topologies differ typically. Namely, we fix $\tau \in (\mathcal{D}-\mathrm{top})[E]$; so, $\tau \in (\mathrm{top})[E]$ for which (E,τ) is a T_1-space and (in this section) we suppose that

$$\mathcal{L} = \mathbf{C}_E[\tau]. \tag{7.1}$$

Under (7.1), we call u/f of the set $\mathbb{F}_0^*(\mathcal{L})$ as closed u/f. Analogously, for MLS of $(\mathcal{L}-\mathrm{link})_0[E]$, under (7.1), we use the term closed MLS. In addition, in our case by (5.1) and (7.1)

$$\mathbf{C}_E[\tau] \in (\mathrm{LAT})_0[E] \cap \widetilde{\pi}^0[E]. \tag{7.2}$$

Of course, $\mathbf{C}_E[\tau] \in (\mathcal{D}-\mathrm{clos})[E]$. By (7.2) we have the separable lattice (7.1). Indeed, $\{x\} \in \mathbf{C}_E[\tau]$ under $x \in E$ (really, by (1.19) (E,τ) is a T_1-space). So, in our case, by (2.3) and (5.2)

$$(\mathcal{L}-\mathrm{triv})[x] \in \mathbb{F}_0^*(\mathbf{C}_E[\tau]) \quad \forall\, x \in E. \tag{7.3}$$

Of course, by (7.1) and (7.2) we can use statements of Section 5. In particular, by (5.6), (7.1), and (7.2)

$$\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(F),\mathbf{T}_{\mathbf{C}_E[\tau]}^*[E]\big) = \mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(F),\mathbf{T}_{\mathbf{C}_E[\tau]}^0[E]\big) = \Phi_{\mathbf{C}_E[\tau]}(F)\ \ \forall\, F \in \mathbf{C}_E[\tau]. \tag{7.4}$$

We recall [17, (4.7)] that $\forall L_1 \in \mathcal{L} \ \ \forall L_2 \in \mathcal{L}$

$$(L_1 \subset L_2) \Longleftrightarrow \big(\Phi_{\mathcal{L}}(L_1) \subset \Phi_{\mathcal{L}}(L_2)\big). \tag{5.7}$$

As an obvious corollary, for $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$

$$(L_1 = L_2) \Longleftrightarrow \big(\Phi_{\mathcal{L}}(L_1) = \Phi_{\mathcal{L}}(L_2)\big).$$

Proposition 7. *If $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$, then*

$$(L_1 \subset L_2) \Longleftrightarrow \big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] \subset (\mathcal{L}-\mathrm{link})^0[E|\,L_2]\big). \tag{5.8}$$

P r o o f. By [5, (2.15)] we have implication

$$(L_1 \subset L_2) \Longrightarrow \big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] \subset (\mathcal{L}-\mathrm{link})^0[E|\,L_2]\big). \tag{5.9}$$

Let $(\mathcal{L}-\mathrm{link})^0[E|\,L_1] \subset (\mathcal{L}-\mathrm{link})^0[E|\,L_2]$. We prove that $L_1 \subset L_2$. Indeed, suppose the contrary: let

$$L_1 \setminus L_2 \neq \emptyset. \tag{5.10}$$

With employment of (5.10), we choose $x_* \in L_1 \setminus L_2$. Then, $(\mathcal{L}-\mathrm{triv})[x_*] \in \mathbb{F}_0^*(\mathcal{L})$ and, in particular (see (2.14)),

$$(\mathcal{L}-\mathrm{triv})[x_*] \in (\mathcal{L}-\mathrm{link})_0[E]. \tag{5.11}$$

In addition, by the choice of x_* we obtain (see Section 2) that $L_1 \in (\mathcal{L}-\mathrm{triv})[x_*]$. Then, by (3.8) and (5.11)

$$(\mathcal{L}-\mathrm{triv})[x_*] \in (\mathcal{L}-\mathrm{link})^0[E|\,L_1].$$

Therefore, $(\mathcal{L}-\mathrm{triv})[x_*] \in (\mathcal{L}-\mathrm{link})^0[E|\,L_2]$ (we use our supposition). Using (3.8), we obtain that $L_2 \in (\mathcal{L}-\mathrm{triv})[x_*]$ and, as a corollary, $x_* \in L_2$. But, this inclusion contradicts to the choice of x_* (recall that $x_* \notin L_2$). The obtained contradiction proves the required inclusion $L_1 \subset L_2$. So, implication

$$\big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] \subset (\mathcal{L}-\mathrm{link})^0[E|\,L_2]\big) \Longrightarrow (L_1 \subset L_2) \tag{5.12}$$

is established. From (5.9) and (5.12), we obtain (5.8). □

Corollary 1. *If $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$, then*

$$(L_1 = L_2) \Longleftrightarrow \big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] = (\mathcal{L}-\mathrm{link})^0[E|\,L_2]\big).$$

The corresponding proof is obvious (see Proposition 7). So, mapping

$$L \longmapsto (\mathcal{L}-\mathrm{link})^0[E|\,L] : \mathcal{L} \longrightarrow \mathfrak{C}_0^*[E;\mathcal{L}]$$

is a bijection from $\mathcal{L}$ onto $\mathfrak{C}_0^*[E;\mathcal{L}]$ (see (3.10) and Corollary 1). We note that from (5.6) the next density property follows:

$$\mathrm{cl}\big((\mathcal{L}-\mathrm{triv})[\cdot]^1(E), \mathbf{T}_{\mathcal{L}}^*[E]\big) = \mathrm{cl}\big((\mathcal{L}-\mathrm{triv})[\cdot]^1(E),\, \mathbf{T}_{\mathcal{L}}^0[E]\big) = \mathbb{F}_0^*(\mathcal{L}); \tag{5.13}$$

in (5.13), we use the obvious equality $\Phi_{\mathcal{L}}(E) = \mathbb{F}_0^*(\mathcal{L})$.

The corresponding proof (see [5]) is immediate combination of (4.10), (4.14), and (4.15) (see [13, 6.2]).

We note the following obvious property (see (3.8) and definitions of Section 2)

$$\Phi_{\mathcal{L}}(L) = (\mathcal{L} - \mathrm{link})^0[E|\, L] \cap \mathbb{F}_0^*(\mathcal{L}) \;\; \forall\, L \in \mathcal{L}. \tag{4.16}$$

Therefore, by (2.4), (3.10), and (4.16) we obtain the equality

$$(\mathbb{UF})[E; \mathcal{L}] = \mathfrak{C}_0^*[E; \mathcal{L}]|_{\mathbb{F}_0^*(\mathcal{L})}.$$

As a corollary, the following important property (see [5]) is realized:

$$\mathbf{T}_{\mathcal{L}}^*[E] = \mathbb{T}_*(E|\, \mathcal{L})|_{\mathbb{F}_0^*(\mathcal{L})}. \tag{4.17}$$

From (4.17), we obtain the next statement: (2.5) is a subspace of the TS (4.3). So, by (3.30) and (4.17)

$$\big(\mathbf{T}_{\mathcal{L}}^0[E] = \mathbb{T}_0(E|\, \mathcal{L})|_{\mathbb{F}_0^*(\mathcal{L})}\big) \& \big(\mathbf{T}_{\mathcal{L}}^*[E] = \mathbb{T}_*(E|\, \mathcal{L})|_{\mathbb{F}_0^*(\mathcal{L})}\big). \tag{4.18}$$

In (4.18), we have the natural connection for topological equipments of the spaces of MLS and u/f. In addition, by [5, Proposition 6.5]

$$\mathbb{T}_0(E|\, \mathcal{L}) \subset \mathbb{T}_*(E|\, \mathcal{L}). \tag{4.19}$$

So, by (4.19) we obtain the following BTS

$$\big((\mathcal{L} - \mathrm{link})_0[E], \mathbb{T}_0(E|\, \mathcal{L}), \mathbb{T}_*(E|\, \mathcal{L})\big). \tag{4.20}$$

Of course, by (4.18) we can consider BTS (2.12) as a subspace of BTS (4.20).

5. Ultrafilters of separable lattice of sets

In present section, we suppose that

$$\mathcal{L} \in (\mathrm{LAT})_0[E] \cap \widetilde{\pi}^0[E]. \tag{5.1}$$

By (5.1) we obtain the case of separable lattice. Using (2.3) and (5.1), we obtain that

$$(\mathcal{L} - \mathrm{triv})[x] \in \mathbb{F}_0^*(\mathcal{L}) \;\; \forall\, x \in E. \tag{5.2}$$

By (5.2) we can introduce operator

$$x \longmapsto (\mathcal{L} - \mathrm{triv})[x] : \, E \longrightarrow \mathbb{F}_0^*(\mathcal{L}) \tag{5.3}$$

denoted by $(\mathcal{L} - \mathrm{triv})[\cdot]$. Of course, (5.3) is an immersion of E into $\mathbb{F}_0^*(\mathcal{L})$. Therefore, we can consider sets-images

$$(\mathcal{L} - \mathrm{triv})[\cdot]^1(A) \stackrel{\triangle}{=} \{(\mathcal{L} - \mathrm{triv})[x] : \, x \in A\} \in \mathcal{P}\big(\mathbb{F}_0^*(\mathcal{L})\big) \quad \forall\, A \in \mathcal{P}(E). \tag{5.4}$$

We note that by (2.11) and (5.4) the inclusions

$$\mathrm{cl}\big((\mathcal{L} - \mathrm{triv})[\cdot]^1(A), \mathbf{T}_{\mathcal{L}}^*[E]\big) \subset \mathrm{cl}\big((\mathcal{L} - \mathrm{triv})[\cdot]^1(A), \mathbf{T}_{\mathcal{L}}^0[E]\big) \quad \forall\, A \in \mathcal{P}(E) \tag{5.5}$$

are realized. By [5, Proposition 6.6] we have the system of equalities

$$\mathrm{cl}\big((\mathcal{L} - \mathrm{triv})[\cdot]^1(L), \mathbf{T}_{\mathcal{L}}^*[E]\big) = \mathrm{cl}\big((\mathcal{L} - \mathrm{triv})[\cdot]^1(L), \mathbf{T}_{\mathcal{L}}^0[E]\big) = \Phi_{\mathcal{L}}(L) \quad \forall\, L \in \mathcal{L}. \tag{5.6}$$

It is easy proved that $\forall L_1 \in \mathcal{L} \ \ \forall L_2 \in \mathcal{L}$

$$(L_1 \cap L_2 = \emptyset) \Longleftrightarrow \big((\mathcal{L} - \mathrm{link})^0[E|\, L_1] \cap (\mathcal{L} - \mathrm{link})^0[E|\, L_2] = \emptyset\big); \tag{4.6}$$

in (4.5), we use the property analogous to (3.3). We use (4.6) for verification of separability of the TS (4.2). For this, we introduce the next notion: if $\mathcal{E}_1$ and $\mathcal{E}_2$ are nonempty families, then

$$(\mathrm{Dis})[\mathcal{E}_1; \mathcal{E}_2] \overset{\triangle}{=} \{z \in \mathcal{E}_1 \times \mathcal{E}_2 |\, \mathrm{pr}_1(z) \cap \mathrm{pr}_2(z) = \emptyset\}. \tag{4.7}$$

Of course, in (4.7), we can use arbitrary MLS from $(\mathcal{L} - \mathrm{link})_0[E]$ as $\mathcal{E}_1$ and $\mathcal{E}_2$. Then, by (4.6) and (4.7)

$$\begin{gathered}(\mathcal{L} - \mathrm{link})^0[E|\, \mathrm{pr}_1(z)] \cap (\mathcal{L} - \mathrm{link})^0[E|\, \mathrm{pr}_2(z)] = \emptyset \\ \forall \mathcal{E}_1 \in (\mathcal{L} - \mathrm{link})_0[E] \ \ \forall \mathcal{E}_2 \in (\mathcal{L} - \mathrm{link})_0[E] \ \ \forall z \in (\mathrm{Dis})]\mathcal{E}_1; \mathcal{E}_2].\end{gathered} \tag{4.8}$$

If (X, τ) is TS and $x \in X$, then $N_\tau^0(x) \overset{\triangle}{=} \{G \in \tau |\, x \in G\}$. We confine ourselves to employment of open neighborhoods. Of course, by (3.10) and (4.2) we obtain the following obvious property: if $\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E]$ and $\Sigma \in \mathcal{E}$, then

$$(\mathcal{L} - \mathrm{link})^0[E|\, \Sigma] \in N^0_{\mathbb{T}_*(E|\mathcal{L})}(\mathcal{E}). \tag{4.9}$$

We note that by (3.32), for $\mathcal{E}_1 \in (\mathcal{L} - \mathrm{link})_0[E]$ and $\mathcal{E}_2 \in (\mathcal{L} - \mathrm{link})_0[E] \setminus \{\mathcal{E}_1\}$, the property

$$(\mathrm{Dis})[\mathcal{E}_1; \mathcal{E}_2] \neq \emptyset$$

is realized (see (4.7)). As a corollary, by (4.8) and (4.9) $\forall \mathcal{E}_1 \in (\mathcal{L} - \mathrm{link})_0[E]$ $\forall \mathcal{E}_2 \in (\mathcal{L} - \mathrm{link})_0[E] \setminus \{\mathcal{E}_1\} \ \exists \mathbb{G}_1 \in N^0_{\mathbb{T}_*(E|\mathcal{L})}(\mathcal{E}_1) \ \ \exists \mathbb{G}_2 \in N^0_{\mathbb{T}_*(E|\mathcal{L})}(\mathcal{E}_2):$

$$\mathbb{G}_1 \cap \mathbb{G}_2 = \emptyset. \tag{4.10}$$

So, (4.3) is a T_2-space. Moreover, we note that by (2.15)

$$(\mathcal{L} - \mathrm{link})^0[E|\, L] = \{\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E] |\, L \cap \Sigma \neq \emptyset \ \ \forall \Sigma \in \mathcal{E}\} \quad \forall L \in \mathcal{L}. \tag{4.11}$$

On the other hand, from (4.11) the following property (see [5]) is extracted:

$$(\mathcal{L} - \mathrm{link})^0[E|\, L] \in \mathbb{T}_*(E|\, \mathcal{L}) \cap \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_*(E|\mathcal{L})] \quad \forall L \in \mathcal{L}. \tag{4.12}$$

From (3.10) and (4.12), we obtain that

$$\mathfrak{C}^*_0[E; \mathcal{L}] \subset \mathbb{T}_*(E|\, \mathcal{L}) \cap \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_*(E|\, \mathcal{L})]. \tag{4.13}$$

Using axioms of TS, from (4.13), we obtain that

$$\{\cap\}_\sharp(\mathfrak{C}^*_0[E; \mathcal{L}]) \subset \mathbb{T}_*(E|\, \mathcal{L}) \cap \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_*(E|\mathcal{L})], \tag{4.14}$$

where $\{\cap\}_\sharp(\mathfrak{C}^*_0[E; \mathcal{L}]) \in (\mathrm{BAS})\big[(\mathcal{L} - \mathrm{link})_0[E]\big]$ and by (4.2)

$$\{\cap\}_\sharp(\mathfrak{C}^*_0[E; \mathcal{L}]) \in \big(\mathbb{T}_*(E|\, \mathcal{L}) - \mathrm{BAS}\big)_0\big[(\mathcal{L} - \mathrm{link})_0[E]\big]. \tag{4.15}$$

Proposition 6. *In the form of* (4.3) *a zero-dimensional T_2-space is realized.*

Proposition 4. *If* (3.16) *is T_2-space, then $\mathbb{F}_0^*(\mathcal{L})$ is closed in this space:*

$$\mathbb{F}_0^*(\mathcal{L}) \in \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_0(E|\,\mathcal{L})].$$

In connection with Proposition 4, we note the known property concerning to [4, 4.16] (see too [16, p. 65]).

We note that, for every $\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E]$, the following equality is realized:

$$\bigcap_{\Sigma \in \mathcal{E}} (\mathcal{L} - \mathrm{link})^0[E|\,\Sigma] = \{\mathcal{E}\};$$

as a corollary, by (3.19) we obtain that

$$\{\mathcal{E}\} \in \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_0(E|\,\mathcal{L})].$$

So, we have the following statement of [5].

Proposition 5. *By* (3.16) *a supercompact T_1-space is realized.*

Of course, if (3.16) is a T_2-space, then it is a supercompactum. We note that by the maximality property $\forall\, \mathcal{E}_1 \in (\mathcal{L} - \mathrm{link})_0[E] \;\; \forall\, \mathcal{E}_2 \in (\mathcal{L} - \mathrm{link})_0[E]$

$$(\mathcal{E}_1 \neq \mathcal{E}_2) \Longleftrightarrow \big((\mathcal{E}_1 \setminus \mathcal{E}_2 \neq \emptyset)\&(\mathcal{E}_2 \setminus \mathcal{E}_1 \neq \emptyset)\big).$$

Moreover, it is obvious that $\forall\, \mathcal{E}_1 \in (\mathcal{L} - \mathrm{link})_0[E] \;\; \forall\, \mathcal{E}_2 \in (\mathcal{L} - \mathrm{link})_0[E]$

$$(\mathcal{E}_1 \neq \mathcal{E}_2) \Longleftrightarrow (\exists\, \Sigma_1 \in \mathcal{E}_1 \;\; \exists\, \Sigma_2 \in \mathcal{E}_2 : \; \Sigma_1 \cap \Sigma_2 = \emptyset). \tag{3.32}$$

4. Maximal linked systems as elements of zero-dimensional T_2-space and bitopological structure

In this section, we introduce TS analogous to (2.5). Elements of this new TS are MLS. We recall that by (1.14), (3.8), (3.10), and (3.9)

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p} - \mathrm{BAS})_{\emptyset}\big[(\mathcal{L} - \mathrm{link})_0[E]\big]. \tag{4.1}$$

From (4.1), the obvious property $\{\cap\}_\sharp(\mathfrak{C}_0^*[E;\mathcal{L}]) \in (\mathrm{op} - \mathrm{BAS})_{\emptyset}\big[(\mathcal{L} - \mathrm{link})_0[E]\big]$ follows. As a corollary,

$$\mathbb{T}_*(E|\,\mathcal{L}) \overset{\triangle}{=} \{\cup\}\big(\{\cap\}_\sharp(\mathfrak{C}_0^*[E;\mathcal{L}])\big) \in (\mathrm{top})\big[(\mathcal{L} - \mathrm{link})_0[E]\big]. \tag{4.2}$$

So, by (4.2) we obtain the required TS

$$\big((\mathcal{L} - \mathrm{link})_0[E], \mathbb{T}_*(E|\,\mathcal{L})\big). \tag{4.3}$$

For this TS, by (4.2) we have the inclusion

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p} - \mathrm{BAS})_{\emptyset}^0\big[(\mathcal{L} - \mathrm{link})_0[E]; \mathbb{T}_*(E|\,\mathcal{L})\big]. \tag{4.4}$$

So, we obtain the following statement

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p} - \mathrm{BAS})_{\emptyset}^0\big[(\mathcal{L} - \mathrm{link})_0[E]; \mathbb{T}_*(E|\,\mathcal{L})\big] \cap \big((\mathrm{p}, \mathrm{bin}) - \mathrm{cl}\big)\big[(\mathcal{L} - \mathrm{link})_0[E]; \mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{4.5}$$

We obtain some analog of (2.13). So, the family $\mathfrak{C}_0^*[E;\mathcal{L}]$ «serves» both topology $\mathbb{T}_*(E|\,\mathcal{L})$ and topology $\mathbb{T}_0(E|\,\mathcal{L})$. But, now we focus on consideration of TS (4.3).

is the family of all closed binary subbases. We recall that by (1.40)

$$\Big(\mathbb{T}_0(E|\,\mathcal{L})\in\big((\mathbb{SC})-\mathrm{top}\big)\big[(\mathcal{L}-\mathrm{link})_0[E]\big]\Big)\Leftrightarrow\Big(\big((\mathrm{p},\mathrm{bin})-\mathrm{cl}\big)\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]\neq\emptyset\Big). \quad (3.21)$$

In [5], the following statement was established: $\mathfrak{C}_0^*[E;\mathcal{L}]\in\big((\mathrm{p},\mathrm{bin})-\mathrm{cl}\big)\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]$. From (3.21), we obtain that

$$\mathbb{T}_0(E|\,\mathcal{L})\in\big((\mathbb{SC})-\mathrm{top}\big)\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \quad (3.22)$$

So, (3.16) is a supercompact TS. With employment of (1.39), (3.18), and the above-mentioned property of $\mathfrak{C}_0^*[E;\mathcal{L}]$, we have the following statement:

$$\forall\,\mathcal{C}\in(\mathrm{COV})\big[(\mathcal{L}-\mathrm{link})_0[E]|\,\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]\big]\ \exists\,C_1\in\mathcal{C}\ \exists\,C_2\in\mathcal{C}:\ (\mathcal{L}-\mathrm{link})_0[E]=C_1\cup C_2. \quad (3.23)$$

Of course, for (3.23), we use the property

$$\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]=\mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}\big[\mathfrak{C}_o^*[E;\mathcal{L}]\big] \quad (3.24)$$

(indeed, (3.24) is obvious corollary of (3.18)). We consider (3.15) as a topology of Wallman type. We note two obvious property. Namely, for $\Lambda_1\in\mathbf{C}_E[\mathcal{L}]$ and $\Lambda_2\in\mathbf{C}_E[\mathcal{L}]$

$$(\Lambda_1\cap\Lambda_2=\emptyset)\Rightarrow\big((\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda_1]\cap(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda_2]=\emptyset\big).$$

Moreover, we have the following property of isotonicity: under $\Lambda_1\in\mathbf{C}_E[\mathcal{L}]$ and $\Lambda_2\in\mathbf{C}_E[\mathcal{L}]$

$$(\Lambda_1\subset\Lambda_2)\Rightarrow\big((\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda_1]\subset(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda_2]\big).$$

Now, we consider the corresponding equipment for the set of u/f of the lattice $\mathcal{L}$. For $\Lambda\in\mathbf{C}_E[\mathcal{L}]$, we obtain that

$$\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,\Lambda]\overset{\triangle}{=}(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda]\cap\mathbb{F}_0^*(\mathcal{L})=\{\mathcal{U}\in\mathbb{F}_0^*(\mathcal{L})|\,\exists\,U\in\mathcal{U}:\ U\subset\Lambda\}\in\mathcal{P}\big(\mathbb{F}_0^*(\mathcal{L})\big). \quad (3.25)$$

Of course, $\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,E\setminus L]$ is defined under $L\in\mathcal{L}$. It is obvious that

$$\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]\overset{\triangle}{=}\{\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,\Lambda]:\ \Lambda\in\mathbf{C}_E[\mathcal{L}]\}=\mathbf{C}_{\mathbb{F}_0^*(\mathcal{L})}\big[(\mathbb{UF})[E;\mathcal{L}]\big]. \quad (3.26)$$

In (3.26), the following equality is used: namely, under $L\in\mathcal{L}$, $\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,E\setminus L]=\mathbb{F}_0^*(\mathcal{L})\setminus\Phi_{\mathcal{L}}(L)$. Using simple corollary of (2.8) and (3.26), we obtain that $\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]\in(\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})]$ (see (1.12)). In addition, by (1.13), (2.8), and (3.26)

$$\mathbf{T}^0_{\mathcal{L}}[E]=\{\cup\}(\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]). \quad (3.27)$$

So, we obtain the following property (see [5]): namely,

$$\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]\in(\mathbf{T}^0_{\mathcal{L}}[E]-\mathrm{BAS})_0[\mathbb{F}_0^*(\mathcal{L})]. \quad (3.28)$$

In (3.27) and (3.28), we have analog of (3.15) and (3.17) respectively; in addition, it is useful to note that $\emptyset=\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,\emptyset]\in\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]$ (we use (3.11)) and therefore

$$\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]\in(\mathrm{op}-\mathrm{BAS})_{\emptyset}[\mathbb{F}_0^*(\mathcal{L})].$$

On the other hand, by (3.14), (3.25) and (3.26)

$$\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]=\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]|_{\mathbb{F}_0^*(\mathcal{L})}. \quad (3.29)$$

From (3.29), we obtain the following statement of [5]: (2.10) is a subspace of TS (3.16). Namely

$$\mathbf{T}^0_{\mathcal{L}}[E]=\mathbb{T}_0(E|\,\mathcal{L})|_{\mathbb{F}_0^*(\mathcal{L})}. \quad (3.30)$$

As a corollary, we obtain the useful property: the set $\mathbb{F}_0^*(\mathcal{L})$ is compact in TS (3.16):

$$\mathbb{F}_0^*(\mathcal{L})\in\big(\mathbb{T}_0(E|\,\mathcal{L})-\mathrm{comp}\big)\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \quad (3.31)$$

Now, the following statement is obvious.

(here and later, we follow to [5]). Using (3.8) and (3.9), we obtain that

$$\mathfrak{C}_0^*[E;\mathcal{L}] \stackrel{\triangle}{=} \{(\mathcal{L}-\mathrm{link})^0[E|\,L] : \; L \in \mathcal{L}\} \in \mathcal{P}'\Big(\mathcal{P}\big((\mathcal{L}-\mathrm{link})_0[E]\big)\Big); \tag{3.10}$$

in addition, $\emptyset \in \mathfrak{C}_0^*[E;\mathcal{L}]$ and $(\mathcal{L}-\mathrm{link})_0[E] \in \mathfrak{C}_0^*[E;\mathcal{L}]$. The basic properties of the family (3.10) are considered later. Now, we pass to equipment by topology of the Wallman type. For this, we note that

$$\mathbf{C}_E[\mathcal{L}] = \{E \setminus L : \; L \in \mathcal{L}\} \in (\mathrm{LAT})_0[E]. \tag{3.11}$$

In the form of (3.11), we obtain the lattice dual with respect to $\mathcal{L}$.

Remark 3.1. We recall (see [5]) that $\mathcal{L} = \mathbf{C}_E[\mathcal{L}]$ under $\mathcal{L} \in (\mathrm{alg})[E]$. So, for the particular case, when $(E,\mathcal{L})$ is a measurable space with algebra of sets, the dual lattice (3.11) coincides with $\mathcal{L}$. □

Under $\Lambda \in \mathbf{C}_E[\mathcal{L}]$, we suppose that

$$(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda] \stackrel{\triangle}{=} \{\mathcal{E} \in (\mathcal{L}-\mathrm{link})_0[E]| \;\; \exists\, \Sigma \in \mathcal{E} : \; \Sigma \subset \Lambda\}; \tag{3.12}$$

of course, we can consider that $\Lambda = E \setminus L$, where $L \in \mathcal{L}$. In this connection, we note that

$$(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,E \setminus L] = (\mathcal{L}-\mathrm{link})_0[E] \setminus (\mathcal{L}-\mathrm{link})^0[E|\,L] \quad \forall\, L \in \mathcal{L}. \tag{3.13}$$

Of course, by (3.11) $\emptyset \in \mathbf{C}_E[\mathcal{L}]$ and $E \in \mathbf{C}_E[\mathcal{L}]$; in addition,

$$\big((\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\emptyset] = \emptyset\big) \& \big((\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,E] = (\mathcal{L}-\mathrm{link})_0[E]\big).$$

As a corollary, we obtain that by statements of Section 1

$$\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}] \stackrel{\triangle}{=} \{(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda] : \; \Lambda \in \mathbf{C}_E[\mathcal{L}]\} \in (\mathrm{p}-\mathrm{BAS})_{\emptyset}\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \tag{3.14}$$

As a corollary, in the form of $\{\cap\}_{\sharp}(\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]) \in (\mathrm{op}-\mathrm{BAS})_{\emptyset}\big[(\mathcal{L}-\mathrm{link})_0[E]\big]$, we obtain an open base and

$$\mathbb{T}_0(E|\,\mathcal{L}) \stackrel{\triangle}{=} \{\cup\}\big(\{\cap\}_{\sharp}(\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}])\big) \in (\mathrm{top})\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \tag{3.15}$$

So, we have the following TS

$$\big((\mathcal{L}-\mathrm{link})_0[E],\; \mathbb{T}_0(E|\,\mathcal{L})\big). \tag{3.16}$$

Of course, $\{\cap\}_{\sharp}(\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]) \in \big(\mathbb{T}_0(E|\,\mathcal{L})-\mathrm{BAS}\big)_0\big[(\mathcal{L}-\mathrm{link})_0[E]\big]$ and, as a corollary,

$$\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})^0_{\emptyset}\big[(\mathcal{L}-\mathrm{link})_0[E]; \mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{3.17}$$

In addition, by (3.13) the following equality is realized:

$$\mathfrak{C}_0^*[E;\mathcal{L}] = \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}\big[\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]\big]. \tag{3.18}$$

From (3.17) and (3.18), by duality we obtain (see (1.16)) that

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}\big[(\mathcal{L}-\mathrm{link})_0[E]; \mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{3.19}$$

From (3.17) and (3.19), we have dual construction for TS (3.16). In addition, (3.17) and (3.19) are open and closed subbases of this TS respectively. By (3.18) self these subbases are situated in a duality. Now, we note the statements of [5] connected with supercompactness of TS (3.16). At first, we recall the notion of closed binary subbases. Namely, by (1.38)

$$\begin{gathered}\big((\mathrm{p},\mathrm{bin})-\mathrm{cl}\big)\big[(\mathcal{L}-\mathrm{link})_0[E]; \mathbb{T}_0(E|\,\mathcal{L})\big] = \Big\{\mathfrak{L} \in (\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}\big[(\mathcal{L}-\mathrm{link})_0[E]; \mathbb{T}_0(E|\,\mathcal{L})\big]\,\Big| \\ \bigcap_{\mathbb{L}\in\lambda} \mathbb{L} \neq \emptyset \;\; \forall\, \lambda \in (\mathfrak{L}-\mathrm{link})\big[(\mathcal{L}-\mathrm{link})_0[E]\big]\Big\}\end{gathered} \tag{3.20}$$

3. Maximal linked systems; topology of the Wallman type

We recall that (2.14)–(2.18) are fulfilled under $\mathcal{L} = \mathcal{P}(E)$ (the lattice of all subsets of E). In addition, $(\mathrm{link})[E] = \big(\mathcal{P}(E) - \mathrm{link}\big)[E]$ and $(\mathrm{link})_0[E] = \big(\mathcal{P}(E) - \mathrm{link}\big)_0[E]$ (see (1.34)). As variant of (1.36) and (2.15), we obtain that

$$(\mathrm{link})_0[E] = \{\mathcal{E} \in (\mathrm{link})[E] |\, \forall\, \mathcal{S} \in (\mathrm{link})[E] \ \ (\mathcal{E} \subset \mathcal{S}) \Rightarrow (\mathcal{E} = \mathcal{S})\} = \\ = \{\mathcal{E} \in (\mathrm{link})[E] |\, \forall\, L \in \mathcal{P}(E) \ \ (L \cap \Sigma \neq \emptyset \ \ \forall\, \Sigma \in \mathcal{E}) \Rightarrow (L \in \mathcal{E})\}, \tag{3.1}$$

$(\mathrm{link})_0[E] \neq \emptyset$. From (2.18), we obtain that

$$E \in \mathcal{E} \ \ \forall\, \mathcal{E} \in (\mathrm{link})_0[E]. \tag{3.2}$$

By (2.16) we obtain that

$$\forall\, \mathcal{E}_1 \in (\mathrm{link})[E] \ \exists\, \mathcal{E}_2 \in (\mathrm{link})_0[E] : \ \mathcal{E}_1 \subset \mathcal{E}_2. \tag{3.3}$$

Now, we return to arbitrary fixed lattice (2.6). Using (3.1), we consider one property of MLS for lattice (2.6). But, at first, we note one simple corollary of Proposition 1.

Proposition 2. *The following property takes place:*

$$\mathcal{E} \cap \mathcal{L} \in (\mathcal{L} - \mathrm{link})[E] \quad \forall\, \mathcal{E} \in (\mathrm{link})_0[E]. \tag{3.4}$$

P r o o f. Let $\mathcal{S} \in (\mathrm{link})_0[E]$. Using (2.6), consider the family $\mathcal{S} \cap \mathcal{L}$. By (1.4), (1.6), (2.6) and (3.2) $E \in \mathcal{S} \cap \mathcal{L}$. So, $\mathcal{S} \cap \mathcal{L} \neq \emptyset$ and by Proposition 1 $\mathcal{S} \cap \mathcal{L} \in (\mathcal{L} - \mathrm{link})[E]$. □

Proposition 3. *If* $\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E]$, *then*

$$\exists\, \mathcal{S} \in (\mathrm{link})_0[E] : \ \mathcal{E} = \mathcal{S} \cap \mathcal{L}.$$

P r o o f. Fix $\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E]$. Then, in particular, $\mathcal{E} \in (\mathcal{L} - \mathrm{link})[E]$ and $\forall\, \mathcal{C} \in (\mathcal{L} - \mathrm{link})[E]$

$$(\mathcal{E} \subset \mathcal{C}) \Longrightarrow (\mathcal{E} = \mathcal{C}). \tag{3.5}$$

By (1.35) $\mathcal{E} \in (\mathrm{link})[E]$ and $\mathcal{E} \subset \mathcal{L}$. Then (see (3.3)), for some MLS $\mathcal{V} \in (\mathrm{link})_0[E]$

$$\mathcal{E} \subset \mathcal{V}. \tag{3.6}$$

In addition, by Proposition 2

$$\mathcal{V} \cap \mathcal{L} \in (\mathcal{L} - \mathrm{link})[E]. \tag{3.7}$$

From (3.6), the inclusion $\mathcal{E} \subset \mathcal{V} \cap \mathcal{L}$ is realized. By (3.5) and (3.7) we obtain the equality

$$\mathcal{E} = \mathcal{V} \cap \mathcal{L}.$$

So, $\mathcal{V} \in (\mathrm{link})_0[E] : \ \mathcal{E} = \mathcal{V} \cap \mathcal{L}$. □

We suppose by analogy with [4, 4.10] that

$$(\mathcal{L} - \mathrm{link})^0[E|\, L] \overset{\triangle}{=} \{\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E] |\, L \in \mathcal{E}\} \quad \forall\, L \in \mathcal{L}. \tag{3.8}$$

Of course, we have the following particular cases:

$$\big((\mathcal{L} - \mathrm{link})^0[E|\, \emptyset] = \emptyset\big) \& \big((\mathcal{L} - \mathrm{link})^0[E|\, E] = (\mathcal{L} - \mathrm{link})_0[E]\big) \tag{3.9}$$

By (1.5) and (2.6) we obtain that $\Phi_{\mathcal{L}}(L_1 \cup L_2) \in \mathcal{P}\big(\mathbb{F}_0^*(\mathcal{L})\big)$ is defined under $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$; in addition [6], $\Phi_{\mathcal{L}}(L_1 \cup L_2) = \Phi_{\mathcal{L}}(L_1) \cup \Phi_{\mathcal{L}}(L_2)$. And what is more, in our case (under (2.6))

$$(\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{LAT})_0[\mathbb{F}_0^*(\mathcal{L})]. \tag{2.7}$$

Remark 2.1. From (2.4) and (2.7), the following singularity is noticcable: for $(\mathbb{UF})[E;\mathcal{L}]$, properties of $\mathcal{L}$ are repeated. In this connection, we recall [6, (9.6)]:

$$\big(\mathcal{L} \in (\mathrm{alg})[E]\big) \Longrightarrow \big((\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{alg})[\mathbb{F}_0^*(\mathcal{L})]\big). \qquad \square$$

Returning to general case of (2.6), we note that (see [6, (6.7)])

$$(\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{cl}-\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})]; \tag{2.8}$$

(2.8) permit to define yet one topology. Indeed, by (2.8)

$$\{\cap\}\big((\mathbb{UF})[E;\mathcal{L}]\big) \in (\mathrm{clos})[\mathbb{F}_0^*(\mathcal{L})].$$

As a corollary, we obtain that

$$\mathbf{T}_{\mathcal{L}}^0[E] \triangleq \mathbf{C}_{\mathbb{F}_0^*(\mathcal{L})}\big[\{\cap\}\big((\mathbb{UF})[E;\mathcal{L}]\big)\big] \in (\mathrm{top})[\mathbb{F}_0^*(\mathcal{L})]. \tag{2.9}$$

In addition, topology (2.9) converts [6, Section 6] $\mathbb{F}_0^*(\mathcal{L})$ in a compact T_1-space

$$\big(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^0[E]\big). \tag{2.10}$$

We consider (2.5) as analog of Stone space and (2.10) as analog of Wallman space (the space of Wallman extension). In addition (see [15, Proposition 4.1])

$$\mathbf{T}_{\mathcal{L}}^0[E] \subset \mathbf{T}_{\mathcal{L}}^*[E]. \tag{2.11}$$

With regard to (2.11), we consider triplet

$$\big(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^0[E], \mathbf{T}_{\mathcal{L}}^*[E]\big) \tag{2.12}$$

as a bitopological space (BTS); in this connection, see [9]. We do not discuss inessential differences with constructions of [9] and follow to above-mentioned interpretation of (2.12). So,

$$(\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})] \cap (\mathrm{cl}-\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})] \tag{2.13}$$

generates BTS (2.12). It is useful to note the important particular case; namely, if $\mathcal{L} \in (\mathrm{alg})[E]$, then (2.5) is a zero-dimensional compactum or rather the Stone space.

Maximal linked systems. Now, we consider the families $(\mathcal{L}-\mathrm{link})[E]$ and $(\mathcal{L}-\mathrm{link})_0[E]$. It is obvious that $\mathbb{F}^*(\mathcal{L}) \subset (\mathcal{L}-\mathrm{link})[E]$ and

$$\mathbb{F}_0^*(\mathcal{L}) \subset (\mathcal{L}-\mathrm{link})_0[E]. \tag{2.14}$$

Moreover, easy to check that

$$(\mathcal{L}-\mathrm{link})_0[E] = \{\mathcal{E} \in (\mathcal{L}-\mathrm{link})[E] \,|\, \forall\, L \in \mathcal{L}\ \ (L \cap \Sigma \neq \emptyset\ \ \forall\, \Sigma \in \mathcal{E}) \Longrightarrow (L \in \mathcal{E})\} \tag{2.15}$$

(we use the maximality property). With employment of the Zorn lemma, we obtain that

$$\forall\, \mathcal{E}_1 \in (\mathcal{L}-\mathrm{link})[E]\ \exists\, \mathcal{E}_2 \in (\mathcal{L}-\mathrm{link})_0[E]:\ \mathcal{E}_1 \subset \mathcal{E}_2. \tag{2.16}$$

Finally, we note the following corollary of maximality of MLS: $\forall\, \mathcal{E} \in (\mathcal{L}-\mathrm{link})_0[E]\ \ \forall\, \Sigma \in \mathcal{E}\ \ \forall\, L \in \mathcal{L}$

$$(\Sigma \subset L) \Longrightarrow (L \in \mathcal{E}). \tag{2.17}$$

Therefore, we obtain that

$$E \in \mathcal{E}\ \ \forall\, \mathcal{E} \in (\mathcal{L}-\mathrm{link})_0[E]. \tag{2.18}$$

The property (2.14) is complemented by the following equality:

$$\mathbb{F}_0^*(\mathcal{L}) = \{\mathcal{U} \in (\mathcal{L}-\mathrm{link})_0[E] \,|\, A \cap B \in \mathcal{U}\ \ \forall\, A \in \mathcal{U}\ \ \forall\, B \in \mathcal{U}\} \in \mathcal{P}'\big((\mathcal{L}-\mathrm{link})_0[E]\big).$$

((1.38) is the family of all closed binary subbases of TS (X,τ)); it is obvious that $\forall\,\kappa\in(\mathrm{p-BAS})^0_{\mathrm{cl}}[X;\tau]$

$$\Big(\kappa\in\big((\mathrm{p},\mathrm{bin})-\mathrm{cl}\big)[X;\tau]\Big)\Leftrightarrow\big(\forall\,\mathfrak{C}\in(\mathrm{COV})\big[X|\,\mathbf{C}_X[\kappa]\big]\ \exists\,C_1\in\mathfrak{C}\ \exists\,C_2\in\mathfrak{C}:\ X=C_1\cup C_2\big). \quad (1.39)$$

In addition, we suppose that

$$\big((\mathbb{SC})-\mathrm{top}\big)[X]\overset{\triangle}{=}\big\{\tau\in(\mathrm{top})[X]\,|\,\big((\mathrm{p},\mathrm{bin})-\mathrm{cl}\big)[X;\tau]\neq\emptyset\big\}; \quad (1.40)$$

in addition, $\big((\mathbb{SC})-\mathrm{top}\big)[X]$ is the family of all supercompact topologies on X. Under $\tau\in\big((\mathbb{SC})-\mathrm{top})[X]$, we obtain supercompact TS (X,τ); moreover, if (X,τ) is a T_2-space, then (X,τ) is called supercompactum. Every supercompact TS is compact. Then, under $\tau\in\big((\mathbb{SC})-\mathrm{top})[X]$, in the form of (X,τ), we obtain (in particular) a compact TS.

2. Maximal linked systems and ultrafilters: general properties

In the following, a nonempty set E is fixed. We consider families from $\mathcal{P}'\big(\mathcal{P}(E)\big)$. In addition, we use (1.4)– (1.10).

Filters and ultrafiltres. In the following, we fix $\mathcal{L}\in\pi[E]$ (later, with respect to $\mathcal{L}$, additional conditions will overlap). We consider $(E,\mathcal{L})$ as widely understood measurable space. Then,

$$\mathbb{F}^*(\mathcal{L})\overset{\triangle}{=}\{\mathcal{F}\in\mathcal{P}'(\mathcal{L}\backslash\{\emptyset\})|\,(A\cap B\in\mathcal{F}\ \forall\,A\in\mathcal{F}\ \forall\,B\in\mathcal{F})\&\big(\forall\,F\in\mathcal{F}\ \forall\,L\in\mathcal{L}\ (F\subset L)\Rightarrow(L\in\mathcal{F})\big)\} \quad (2.1)$$

is the family of all filters of $(E,\mathcal{L})$. Maximal filters are called ultrafilters (u/f). Then

$$\begin{gathered}\mathbb{F}^*_0(\mathcal{L})\overset{\triangle}{=}\{\mathcal{U}\in\mathbb{F}^*(\mathcal{L})|\,\forall\,\mathcal{F}\in\mathbb{F}^*(\mathcal{L})\ \ (\mathcal{U}\subset\mathcal{F})\Rightarrow(\mathcal{U}=\mathcal{F})\}=\{\mathcal{U}\in\mathbb{F}^*(\mathcal{L})|\,\forall\,L\in\mathcal{L}\\ (L\cap U\neq\emptyset\ \ \forall\,U\in\mathcal{U})\Rightarrow(L\in\mathcal{U})\}=\{\mathcal{U}\in(\mathrm{Cen})[\mathcal{L}]|\,\forall\,\mathcal{V}\in(\mathrm{Cen})[\mathcal{L}]\ \ (\mathcal{U}\subset\mathcal{V})\Rightarrow(\mathcal{U}=\mathcal{V})\}\end{gathered} \quad (2.2)$$

is the nonempty family of all u/f of $(E,\mathcal{L})$. If $x\in E$, then

$$(\mathcal{L}-\mathrm{triv})[x]\overset{\triangle}{=}\{L\in\mathcal{L}|\,x\in L\}\in\mathbb{F}^*(\mathcal{L})$$

is trivial (fixed) filter corresponding to the point x. It is known [14, (5.9)] that

$$\big((\mathcal{L}-\mathrm{triv})[x]\in\mathbb{F}^*_0(\mathcal{L})\ \ \forall\,x\in E\big)\Leftrightarrow(\mathcal{L}\in\widetilde{\pi}^0[E]). \quad (2.3)$$

We suppose that $\Phi_{\mathcal{L}}(L)\overset{\triangle}{=}\{\mathcal{U}\in\mathbb{F}^*_0(\mathcal{L})|\,L\in\mathcal{U}\}\ \ \forall\,L\in\mathcal{L}$. Then, how easy check,

$$(\mathbb{UF})[E;\mathcal{L}]\overset{\triangle}{=}\{\Phi_{\mathcal{L}}(L):\ L\in\mathcal{L}\}\in\pi[\mathbb{F}^*_0(\mathcal{L})]. \quad (2.4)$$

From (1.11) and (2.4), the inclusion $(\mathbb{UF})[E;\mathcal{L}]\in(\mathrm{BAS})[\mathbb{F}^*_0(\mathcal{L})]$ follows. In addition, topology

$$\mathbf{T}^*_{\mathcal{L}}[E]\overset{\triangle}{=}\{\cup\}\big((\mathbb{UF})[E;\mathcal{L}]\big)=\{\mathbb{G}\in\mathcal{P}\big(\mathbb{F}^*_0(\mathcal{L})\big)|\,\forall\,\mathcal{U}\in\mathbb{G}\ \exists\,U\in\mathcal{U}:\ \Phi_{\mathcal{L}}(U)\subset\mathbb{G}\}\in(\mathrm{top})[\mathbb{F}^*_0(\mathcal{L})]$$

realizes [14] zero-dimensional T_2-space

$$\big(\mathbb{F}^*_0(\mathcal{L}),\mathbf{T}^*_{\mathcal{L}}[E]\big). \quad (2.5)$$

Everywhere in the future, we suppose that

$$\mathcal{L}\in(\mathrm{LAT})_0[E]. \quad (2.6)$$

is known cofinite topology and

$$(\mathrm{FIN})[X] \cup \{X\} = \mathbf{C}_X\big[\tau^0_{(\mathrm{FIN})[X]}[X]\big]$$

is the family of closed sets in this topology.

Example 1.4. Let X, $X \neq \emptyset$, be uncountable set. Consider the family $\omega[X]$ of all no more than countable subsets of X. In addition, $\omega[X] = (\mathrm{count})[X] \cup \{\emptyset\}$, where $(\mathrm{count})[X] \stackrel{\triangle}{=} \{f^1(\mathbb{N}) :\ f \in X^{\mathbb{N}}\}$ under $\mathbb{N} = \{1; 2; \ldots\}$ and $\widetilde{f}^1(\mathbb{N}) = \{\widetilde{f}(k) :\ k \in \mathbb{N}\}$ for $\widetilde{f} \in X^{\mathbb{N}}$. Then, $\omega[X] \in (\downarrow -\mathrm{LAT})^0[X]$. The corresponding proof is similar to previous example. $\square$

Coverings and linked families. Recall that X is a nonempty set. If $\mathcal{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$, then

$$(\mathrm{COV})[X|\,\mathcal{X}] \stackrel{\triangle}{=} \big\{\mathfrak{X} \in \mathcal{P}'(\mathcal{X})|\, X = \bigcup_{\mathbb{X} \in \mathfrak{X}} \mathbb{X}\big\} \tag{1.32}$$

is the family of all coverings of X by sets from $\mathcal{X}$. Let

$$(\mathrm{link})[X] \stackrel{\triangle}{=} \{\mathcal{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)|\, A \cap B \neq \emptyset \ \ \forall A \in \mathcal{X} \ \ \forall B \in \mathcal{X}\}. \tag{1.33}$$

Then, the family of all linked systems of subsets of X is introduced. Moreover, suppose that

$$(\mathrm{link})_0[X] \stackrel{\triangle}{=} \{\mathcal{E} \in (\mathrm{link})[X]|\, \forall \mathcal{S} \in (\mathrm{link})[X] \ \ (\mathcal{E} \subset \mathcal{S}) \Rightarrow (\mathcal{E} = \mathcal{S})\}. \tag{1.34}$$

We obtain the family of all MLS of subsets of X. In the following, we consider MLS containing in a given family. So, under $\mathfrak{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$

$$(\mathfrak{X} - \mathrm{link})[X] \stackrel{\triangle}{=} \{\mathcal{E} \in (\mathrm{link})[X]|\, \mathcal{E} \subset \mathfrak{X}\} \in \mathcal{P}\big((\mathrm{link})[X]\big) \tag{1.35}$$

and by analogy with (1.34)

$$(\mathfrak{X} - \mathrm{link})_0[X] \stackrel{\triangle}{=} \{\mathcal{E} \in (\mathfrak{X} - \mathrm{link})[X]|\, \forall \widetilde{\mathcal{E}} \in (\mathfrak{X} - \mathrm{link})[X] \ \ (\mathcal{E} \subset \widetilde{\mathcal{E}}) \Rightarrow (\mathcal{E} = \widetilde{\mathcal{E}})\}. \tag{1.36}$$

In (1.36), we obtain the family of all MLS containing in the family $\mathfrak{X}$.

Proposition 1. *If* $\mathfrak{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$, $\mathcal{E} \in (\mathrm{link})[X]$, *and* $\mathfrak{X} \cap \mathcal{E} \neq \emptyset$, *then*

$$\mathfrak{X} \cap \mathcal{E} \in (\mathfrak{X} - \mathrm{link})[X]. \tag{1.37}$$

P r o o f. Fix $\mathfrak{X}$ and $\mathcal{E}$ with above-mentioned properties. In particular, $\mathfrak{X} \cap \mathcal{E} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$. Let $U \in \mathfrak{X} \cap \mathcal{E}$ and $V \in \mathfrak{X} \cap \mathcal{E}$. Then, in particular, $U \in \mathcal{E}$ and $V \in \mathcal{E}$. By (1.33) we obtain that $U \cap V \neq \emptyset$. Since the choice of U and V was arbitrary, we have the property

$$\mathfrak{X} \cap \mathcal{E} \in \mathcal{P}'\big(\mathcal{P}(X)\big):\ A \cap B \neq \emptyset \ \ \forall A \in \mathfrak{X} \cap \mathcal{E} \ \ \forall B \in \mathfrak{X} \cap \mathcal{E}.$$

By (1.33) $\mathfrak{X} \cap \mathcal{E} \in (\mathrm{link})[X]$. Then (see (1.35)), (1.37) is fulfilled. $\square$

Supercompactness. If $\tau \in (\mathrm{top})[X]$, then we suppose that

$$\big((\mathrm{p}, \mathrm{bin}) - \mathrm{cl}\big)[X; \tau] \stackrel{\triangle}{=} \big\{\mathfrak{X} \in (\mathrm{p} - \mathrm{BAS})^0_{\mathrm{cl}}[X; \tau]|\, \bigcap_{\mathbb{X} \in \mathcal{X}} \mathbb{X} \neq \emptyset \ \ \forall \mathcal{X} \in (\mathfrak{X} - \mathrm{link})[X]\big\} \tag{1.38}$$

as a corollary, $\mathbb{A} \cap \mathbb{B} \in \mathbf{C}_X[\tau]$. If $\theta \triangleq \tau|_{\mathbb{A}}$, then by transitivity of the operation of passage to a subspace of TS we have the equality

$$\tau|_{\mathbb{A}\cap\mathbb{B}} = \theta|_{\mathbb{A}\cap\mathbb{B}}, \tag{1.28}$$

where $(\mathbb{A}, \theta)$ is a compact TS. By (1.27) $\mathbb{A} \cap \mathbb{B} \in \mathbf{C}_{\mathbb{A}}[\theta]$ and, as a corollary,

$$\mathbb{A} \cap \mathbb{B} \in (\theta - \text{comp})[\mathbb{A}].$$

So, $(\mathbb{A} \cap \mathbb{B}, \theta|_{\mathbb{A}\cap\mathbb{B}})$ is a compact TS. Using (1.28), we obtain that $(\mathbb{A} \cap \mathbb{B}, \tau|_{\mathbb{A}\cap\mathbb{B}})$ is a compact TS. Therefore, $\mathbb{A} \cap \mathbb{B} \in (\tau - \text{comp})[X]$. Since the choice of $\mathbb{A}$ and $\mathbb{B}$ was arbitrary, it is established that $\forall\, A \in (\tau - \text{comp})[X] \ \ \forall\, B \in (\tau - \text{comp})[X]$

$$\big(A \cup B \in (\tau - \text{comp})[X]\big) \& \big(A \cap B \in (\tau - \text{comp})[X]\big). \tag{1.29}$$

So, by (1.5) and (1.29) we obtain that $(\tau - \text{comp})[X] \in (\text{LAT})[X]$. We recall (1.26). Finally, let $\mathcal{T} \in \mathcal{P}'\big((\tau - \text{comp})[X]\big)$. Then, in particular, $\mathcal{T} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$ and we have the set

$$\mathbb{T} \triangleq \bigcap_{T \in \mathcal{T}} T \in \mathcal{P}(X). \tag{1.30}$$

By separability of (X, τ) $\mathcal{T} \subset \mathbf{C}_X[\tau]$ and (see (1.30)) $\mathbb{T} \in \mathbf{C}_X[\tau]$. In addition, $\mathcal{T} \neq \emptyset$. Choose $\mathbf{T} \in \mathcal{T}$; then $\mathbf{T} \in (\tau - \text{comp})[X]$ and $\mathbb{T} \in \mathcal{P}(\mathbf{T})$. We note that $\mathbf{t} \triangleq \tau|_{\mathbf{T}} \in (\text{top})[\mathbf{T}]$ and TS $(\mathbf{T}, \mathbf{t})$ is a compactum. In addition, $\mathbb{T} \in \mathbf{C}_{\mathbf{T}}[\mathbf{t}]$ (indeed, $(\mathbf{T}, \mathbf{t})$ is a closed subspace of (X, τ)). As a corollary, $\mathbb{T} \in (\mathbf{t} - \text{comp})[\mathbf{T}]$; therefore, $\mathbf{t}|_{\mathbb{T}}$ realizes compactum $(\mathbb{T}, \mathbf{t}|_{\mathbb{T}})$. But, by transitivity we obtain that $\mathbf{t}|_{\mathbb{T}} = \tau|_{\mathbb{T}}$. So, $(\mathbb{T}, \tau|_{\mathbb{T}})$ is compactum; as a corollary $\mathbb{T} \in (\tau - \text{comp})[X]$. Since the choice of $\mathcal{T}$ was arbitrary, we establish (see (1.30)) that

$$\bigcap_{K \in \mathcal{C}} K \in (\tau - \text{comp})[X] \quad \forall\, \mathcal{C} \in \mathcal{P}'\big((\tau - \text{comp})[X]\big). \tag{1.31}$$

Therefore (see (1.17), (1.26), (1.29), and (1.31)), we obtain (1.24).

Example 1.3. Consider the case of infinite set X and suppose that $(\text{FIN})[X] \triangleq \text{Fin}(X) \cup \{\emptyset\}$ (the family of all finite subsets of X.) Of course, in our case

$$X \notin (\text{FIN})[X].$$

We show that $(\text{FIN})[X] \in (\downarrow -\text{LAT})^0[X]$. Indeed, $(\text{FIN})[X] \in (\text{LAT})[X]$ by obvious properties of finite sets. Moreover, $\{x\} \in (\text{FIN})[X] \ \ \forall\, x \in X$. Let $\mathcal{F} \in \mathcal{P}'\big((\text{FIN})[X]\big)$. Then, $\mathcal{F} \neq \emptyset$ and $\mathcal{F} \subset (\text{FIN})[X]$. We choose $\mathbb{F} \in \mathcal{F}$. Then, in particular, $\mathbb{F} \in (\text{FIN})[X]$. Since

$$\mathbf{F} \triangleq \bigcap_{F \in \mathcal{F}} F \subset \mathbb{F},$$

we have the obvious inclusion $\mathbf{F} \in (\text{FIN})[X]$. Since the choice of $\mathcal{F}$ was arbitrary, we obtain that

$$\bigcap_{H \in \mathcal{H}} H \in (\text{FIN})[X] \quad \forall\, \mathcal{H} \in \mathcal{P}'\big((\text{FIN})[X]\big).$$

So, the required property $(\text{FIN})[X] \in (\downarrow -\text{LAT})^0[X]$ is established. Now, we note that

$$\tau^0_{(\text{FIN})[X]}[X] = \mathbf{C}_X\big[(\text{FIN})[X]\big] \cup \{\emptyset\} \in (\text{top})[X]$$

realizes the initial lattice $\mathcal{L} \cup \{X\}$ as the lattice $\mathbf{C}_X\big[\tau^0_{\mathcal{L}}[X]\big]$ of closed sets in T_1-space:

$$\mathcal{L} \cup \{X\} = \mathbf{C}_X\big[\tau^0_{\mathcal{L}}[X]\big]; \tag{1.21}$$

in addition, $(X, \tau^0_{\mathcal{L}}[X])$ is not T_2-space and

$$\tau^0_{\mathcal{L}}[X] \neq \mathcal{P}(X). \tag{1.22}$$

Recall that $\mathcal{L} \cup \{X\} \in (\mathrm{LAT})_0[X] \ \ \forall \mathcal{L} \in (\downarrow -\mathrm{LAT})^0[X]$. Now, we consider some examples.

Example 1.1. Suppose that X is equipped with a pseudometric

$$\rho\colon\ X \times X \to [0, \infty[$$

(here, $[0, \infty[\stackrel{\triangle}{=} \{\xi \in \mathbb{R} |\, 0 \leqslant \xi\}$, where $\mathbb{R}$ is real line); so, (X, ρ) is a pseudometric space, $X \neq \emptyset$. Let

$$B_\rho(X, \varepsilon) \stackrel{\triangle}{=} \{y \in X |\, \rho(x, y) \leqslant \varepsilon\} \ \ \forall x \in X \ \ \forall \varepsilon \in [0, \infty[.$$

We suppose that

$$\mathfrak{B}^\sharp(X, \rho) \stackrel{\triangle}{=} \{H \in \mathcal{P}(X) |\, \exists x \in X \ \exists \varepsilon \in [0, \infty[\colon\ H \subset B_\rho(x, \varepsilon)\} =$$

$$= \{H \in \mathcal{P}(X) |\, \exists x \in X \ \exists \varepsilon \in]0, \infty[\colon\ H \subset B_\rho(x, \varepsilon)\},$$

where $]0, \infty[\stackrel{\triangle}{=} \{\xi \in \mathbb{R} |\, 0 < \xi\}$. Of course, $\mathfrak{B}^\sharp(X, \rho)$ is the family of ρ-bounded subsets of X.

We suppose that $X \notin \mathfrak{B}^\sharp(X, \rho)$. So, the pseudometric ρ is unbounded (in particular, real line $\mathbb{R}$ with the metric-modulus can be used as (X, ρ)). Then,

$$\mathfrak{B}^\sharp(X, \rho) \in (\downarrow -\mathrm{LAT})^0[X]. \tag{1.23}$$

The proof of (1.23) is obvions (see (1.17)). We note only that $\mathfrak{B}^\sharp(X, \rho) = \{\cap\}\big(\mathfrak{B}^\sharp(X, \rho)\big)$. $\square$

Example 1.2. Fix a topology $\tau \in (\mathrm{top})[X]$ for which (X, τ) is a T_2-space (of course, $X \neq \emptyset$). We suppose that

$$X \notin (\tau - \mathrm{comp})[X].$$

So, T_2-space (X, τ) is noncompact. Then

$$(\tau - \mathrm{comp})[X] \in (\downarrow -\mathrm{LAT})^0[X]. \tag{1.24}$$

We consider the scheme of the proof of (1.24). In addition, we recall some known properties. So, at first, we show that

$$(\tau - \mathrm{comp})[X] \in (\mathrm{LAT})[X] \tag{1.25}$$

(we check this understandable property). We recall that $\emptyset \in (\tau - \mathrm{comp})[X]$ and

$$\{x\} \in (\tau - \mathrm{comp})[X] \quad \forall x \in X. \tag{1.26}$$

So, $(\tau - \mathrm{comp})[X] \in \mathcal{P}'\big(\mathcal{P}(X)\big)$. Let $\mathbb{A} \in (\tau - \mathrm{comp})[X]$ and $\mathbb{B} \in (\tau - \mathrm{comp})[X]$. Then $\mathbb{A} \cup \mathbb{B} \in (\tau - \mathrm{comp})[X]$ by definition of the compactness property. Consider $\mathbb{A} \cap \mathbb{B}$. By separability of (X, τ) we have that

$$(\mathbb{A} \in \mathbf{C}_X[\tau]) \& (\mathbb{B} \in \mathbf{C}_X[\tau]); \tag{1.27}$$

Recall following useful duality relations:

$$\big(\mathbf{C}_X[\mathcal{B}] \in (\mathrm{op}-\mathrm{BAS})_{\emptyset}[X] \ \ \forall\, \mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[X]\big)\&\big(\mathbf{C}_X[\mathfrak{B}] \in (\mathrm{cl}-\mathrm{BAS})[X] \\ \forall\, \mathfrak{B} \in (\mathrm{op}-\mathrm{BAS})_{\emptyset}[X]\big). \tag{1.12}$$

We note also [6, (1.20)] and some simple corollaries of [6, (1.17)]:

$$\big(\mathbf{C}_X[\{\cap\}(\mathcal{B})] = \{\cup\}(\mathbf{C}_X[\mathcal{B}]) \in (\mathrm{top})[X] \ \ \forall\, \mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[X]\big)\&\big(\{\cap\}(\mathbf{C}_X[\mathfrak{B}]) = \\ \mathbf{C}_X[\{\cup\}(\mathfrak{B})] \in (\mathrm{clos})[X] \ \ \forall\, \mathfrak{B} \in (\mathrm{op}-\mathrm{BAS})_{\emptyset}[X]\big). \tag{1.13}$$

In connection with (1.12) and (1.13), it is useful to note that under $\beta \in (\mathrm{BAS})[X]$

$$\beta \cup \{\emptyset\} \in (\mathrm{op}-\mathrm{BAS})_{\emptyset}[X] : \ \{\cup\}(\beta) = \{\cup\}(\beta \cup \{\emptyset\}).$$

Now, consider some analogs concerning to subbases. In particular,

$$(\mathrm{p}-\mathrm{BAS})_{\emptyset}[X] = \{\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})[X] \,|\, \emptyset \in \{\cap\}_{\sharp}(\mathfrak{X})\}. \tag{1.14}$$

In terms of (1.14), we obtain the next analog of (1.12):

$$\big(\mathbf{C}_X[\mathfrak{X}] \in (\mathrm{p}-\mathrm{BAS})_{\emptyset}[X] \ \ \forall\, \mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X]\big)\& \\ \big(\mathbf{C}_X[\mathcal{X}] \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X] \ \ \forall\, \mathcal{X} \in (\mathrm{p}-\mathrm{BAS})_{\emptyset}[X]\big). \tag{1.15}$$

As a corollary, from (1.15), it follows that $\forall\, \tau \in (\mathrm{top})[X]$

$$\big(\mathbf{C}_X[\mathfrak{X}] \in (\mathrm{p}-\mathrm{BAS})^0_{\emptyset}[X;\tau] \ \ \forall\, \mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}[X;\tau]\big)\& \\ \big(\mathbf{C}_X[\mathcal{X}] \in (\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}[X;\tau] \ \ \forall\, \mathcal{X} \in (\mathrm{p}-\mathrm{BAS})^0_{\emptyset}[X;\tau]\big). \tag{1.16}$$

A special family of lattices. By (1.10) we can consider lattices from $(\mathrm{LAT})[X]$. Now, we introduce the family

$$(\downarrow -\mathrm{LAT})^0[X] \triangleq \big\{\mathcal{L} \in (\mathrm{LAT})[X] \,|\, (X \notin \mathcal{L})\&(\{x\} \in \mathcal{L} \ \ \forall\, x \in X)\&\big(\bigcap_{L \in \mathcal{L}'} L \in \mathcal{L} \ \ \forall\, \mathcal{L}' \in \mathcal{P}'(\mathcal{L})\big)\big\}. \tag{1.17}$$

It is possible to consider elements of (1.17) as lattices of «small» subsets of X. It is obvious that

$$\mathcal{L} \cup \{X\} \in (\mathrm{clos})[X] \quad \forall\, \mathcal{L} \in (\downarrow -\mathrm{LAT})^0[X]. \tag{1.18}$$

The relation (1.18) assumes an amplification. For this, we introduce

$$\big((\mathcal{D}-\mathrm{top})[X] \triangleq \{\tau \in (\mathrm{top})[X] \,|\, \{x\} \in \mathbf{C}_X[\tau] \ \ \forall\, x \in X\}\big)\& \\ \big((\mathcal{D}-\mathrm{clos})[X] \triangleq \{\mathcal{F} \in (\mathrm{clos})[X] \,|\, \{x\} \in \mathcal{F} \ \ \forall\, x \in X\}\big); \tag{1.19}$$

of course, under $\mathbf{t} \in (\mathcal{D}-\mathrm{top})[X]$, in the form of $(X, \mathbf{t})$, we have a T_1-space. In addition, open and closed topologies from (1.19) are situated in the natural duality. From (1.17) and (1.19), we obtain that

$$\mathcal{L} \cup \{X\} \in (\mathcal{D}-\mathrm{clos})[X] \quad \forall\, \mathcal{L} \in (\downarrow -\mathrm{LAT})^0[X]. \tag{1.20}$$

So, under $\mathcal{L} \in (\downarrow -\mathrm{LAT})^0[X]$, we obtain that

$$\tau^0_{\mathcal{L}}[X] \triangleq \mathbf{C}_X[\mathcal{L} \cup \{X\}] = \mathbf{C}_X[\mathcal{L}] \cup \{\emptyset\} \in (\mathcal{D}-\mathrm{top})[X]$$

is the family of all bases of TS (X,τ). In addition,

$$(\mathrm{p}-\mathrm{BAS})[X] \stackrel{\triangle}{=} \{\mathfrak{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \,|\, \{\cap\}_\sharp(\mathfrak{X}) \in (\mathrm{BAS})[X]\} = \big\{\mathfrak{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \,|\, X = \bigcup_{\mathbb{X}\in\mathfrak{X}} \mathbb{X}\big\}$$

is the family of all open subbases of topologies on X. For any $\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})[X]$, we obtain that $\{\cup\}\big(\{\cap\}_\sharp(\mathfrak{X})\big) \in (\mathrm{top})[X]$. Finally, under $\tau \in (\mathrm{top})[X]$, we suppose that

$$(\mathrm{p}-\mathrm{BAS})_0[X;\tau] \stackrel{\triangle}{=} \{\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})[X] \,|\, \{\cap\}_\sharp(\mathfrak{X}) \in (\tau-\mathrm{BAS})_0[X]\};$$

so, we obtain the family of all open subbases of TS (X,τ). It is useful to introduce one auxiliary construction of [6]:

$$(\mathrm{op}-\mathrm{BAS})_\emptyset[X] \stackrel{\triangle}{=} \{\mathcal{B} \in (\mathrm{BAS})[X] \,|\, \emptyset \in \mathcal{B}\};$$

moreover, it is logical to consider the following family:

$$(\mathrm{p}-\mathrm{BAS})_\emptyset[X] \stackrel{\triangle}{=} \{\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})[X] \,|\, \{\cap\}_\sharp(\mathfrak{X}) \in (\mathrm{op}-\mathrm{BAS})_\emptyset[X]\}.$$

If $\tau \in (\mathrm{top})[X]$, then we suppose that

$$(\mathrm{p}-\mathrm{BAS})^0_\emptyset[X;\tau] \stackrel{\triangle}{=} \{\mathcal{X} \in (\mathrm{p}-\mathrm{BAS})_0[X;\tau] \,|\, \emptyset \in \{\cap\}_\sharp(\mathcal{X})\},$$

$$(\mathrm{p}-\mathrm{BAS})^0_\emptyset[X;\tau] \subset (\mathrm{p}-\mathrm{BAS})_\emptyset[X].$$

Of course, under $\mathcal{B} \in (\mathrm{BAS})[X]$, we obtain that $\mathcal{B} \cup \{\emptyset\} \in (\mathrm{op}-\mathrm{BAS})_\emptyset[X]$ and $\{\cup\}(\mathcal{B} \cup \{\emptyset\}) = \{\cup\}(\mathcal{B})$. So, we were introduce an unessential transformation of open bases; the goal of such transformation was indicated in [6, §1].

Now, we consider closed bases and subbases. Let

$$(\mathrm{cl}-\mathrm{BAS})[X] \stackrel{\triangle}{=} \big\{\mathcal{B} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \,|\, (X \in \mathcal{B})\&(\bigcap_{B\in\mathcal{B}} B = \emptyset)\&$$

$$\&\big(\forall B_1 \in \mathcal{B}\ \ \forall B_2 \in \mathcal{B}\ \ \forall x \in X \setminus (B_1 \cup B_2)\ \ \exists B_3 \in \mathcal{B} :\ (B_1 \cup B_2 \subset B_3)\&(x \notin B_3)\big)\big\};$$

so, we introduce the family of all closed bases of topologies on X. Of course, $\{\cap\}(\mathfrak{B}) \in (\mathrm{clos})[X]$ for $\mathfrak{B} \in (\mathrm{cl}-\mathrm{BAS})[X]$. Under $\tau \in (\mathrm{top})[X]$, we suppose that

$$(\mathrm{cl}-\mathrm{BAS})_0[X;\tau] \stackrel{\triangle}{=} \{\mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[X] \,|\, \mathbf{C}_X[\tau] = \{\cap\}(\mathcal{B})\};$$

then, the family of all closed bases of TS (X,τ) is defined. Now, we introduce the family of all closed subbases of topologies on X :

$$(\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X] \stackrel{\triangle}{=} \{\mathcal{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \,|\, \{\cup\}_\sharp(\mathcal{X}) \in (\mathrm{cl}-\mathrm{BAS})[X]\}.$$

Respectively, in the form

$$(\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}[X;\tau] \stackrel{\triangle}{=} \{\mathcal{X} \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X] \,|\, \{\cup\}_\sharp(\mathcal{X}) \in (\mathrm{cl}-\mathrm{BAS})_0[X;\tau]\},$$

we obtain the family of all closed subbases of TS (X,τ). In addition,

$$\Big(\{\cup\}\big(\{\cap\}_\sharp(\mathfrak{S})\big) \in (\mathrm{top})[X]\ \ \forall \mathfrak{S} \in (\mathrm{p}-\mathrm{BAS})[X]\Big)\&$$
$$\Big(\{\cap\}\big(\{\cup\}_\sharp(\mathcal{S})\big) \in (\mathrm{clos})[X]\ \ \forall \mathcal{S} \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X]\Big).$$

of all lattices of subsets of I with «zero» and «unit». In addition, by

$$(\mathrm{alg})[I] \triangleq \{\mathcal{A} \in \pi[I] |\, I \setminus A \in \mathcal{A} \ \ \forall\, A \in \mathcal{A}\} \tag{1.7}$$

the family of all algebras of subsets of I is defined. Moreover, by

$$(\mathrm{top})[I] \triangleq \{\tau \in \pi[I] |\, \bigcup_{G \in \mathcal{G}} G \in \tau \ \ \forall\, \mathcal{G} \in \mathcal{P}'(\tau)\} = \{\tau \in (\mathrm{LAT})_0[I] |\, \bigcup_{G \in \mathcal{G}} G \in \tau \ \ \forall\, \mathcal{G} \in \mathcal{P}'(\tau)\} \tag{1.8}$$

and (analogously)

$$\begin{gathered}(\mathrm{clos})[I] \triangleq \big\{\mathcal{F} \in \mathcal{P}'\big(\mathcal{P}(I)\big) \,|\, (\emptyset \in \mathcal{F})\&(I \in \mathcal{F})\&(A \cup B \in \mathcal{F} \ \ \forall\, A \in \mathcal{F} \ \ \forall\, B \in \mathcal{F})\& \\ \big(\bigcap_{F \in \mathcal{F}'} F \in \mathcal{F} \ \ \forall\, \mathcal{F}' \in \mathcal{P}'(\mathcal{F})\big)\big\} = \big\{\mathcal{F} \in (\mathrm{LAT})_0[I] |\, \bigcap_{F \in \mathcal{F}'} F \in \mathcal{F} \ \ \forall\, \mathcal{F}' \in \mathcal{P}'(\mathcal{F})\big\}\end{gathered} \tag{1.9}$$

we define the families of all open and closed [11] topologies on I respectively. So, by (1.7)– (1.9) we obtain many useful examples of lattices of the family (1.6). Yet one particular case of a lattice of subsets of I is connected with σ-topologies of A.D. Alexandroff [12]:

$$(\sigma - \mathrm{top})[I] \triangleq \{\tau \in \pi[I] |\, \bigcup_{k \in \mathbb{N}} G_k \in \tau \ \ \forall\, (G_k)_{k \in \mathbb{N}} \in \tau^{\mathbb{N}}\} \subset (\mathrm{LAT})_0[I],$$

where as usually $\mathbb{N} \triangleq \{1; 2; \ldots\}$. Of course, under $\mathcal{A} \in (\mathrm{alg})[I]$, in the form of $(I, \mathcal{A})$, we obtain a measurable space with algebra of sets. If $\tau \in (\mathrm{top})[I]$, then (I, τ) is a topological space (TS). In addition, we use the notions T_1- and T_2-space (see [13, Ch.1]). Moreover, we use compactness [13, Ch.3] and other notions relating to general topology; see [13]. In particular, under $\tau \in (\mathrm{top})[I]$, by $(\tau - \mathrm{comp})[I]$ the family of all compact in (I, τ) subsets of I is denoted; $(\tau - \mathrm{comp})[I] \in \mathcal{P}'\big(\mathcal{P}(I)\big)$. We note the obvious property

$$\mathcal{L} \cup \{I\} \in (\mathrm{LAT})_0[I] \quad \forall\, \mathcal{L} \in (\mathrm{LAT})[I]. \tag{1.10}$$

Of course, in (1.10) we have an insignificant transformation of initial lattice.

Let

$$\widetilde{\pi}^0[I] \triangleq \{\mathcal{L} \in \pi[I] |\, \forall\, L \in \mathcal{L} \ \ \forall\, x \in I \setminus L \ \ \exists\, \Lambda \in \mathcal{L} :\ (x \in \Lambda)\&(\Lambda \cap L = \emptyset)\}.$$

Moreover, let

$$(\mathrm{Cen})[\mathcal{L}] \triangleq \{\mathcal{Z} \in \mathcal{P}'(\mathcal{L}) |\, \bigcap_{Z \in \mathcal{K}} Z \neq \emptyset \ \ \forall\, \mathcal{K} \in \mathrm{Fin}(\mathcal{Z})\} \quad \forall\, \mathcal{L} \in \pi[I].$$

Bases and subbases. For brevity of desingnations, until end of this section, we fix a nonempty set X and use (1.1). Then,

$$\begin{gathered}(\mathrm{BAS})[X] \triangleq \big\{\mathcal{B} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \,|\, \big(X = \bigcup_{B \in \mathcal{B}} B\big)\&\big(\forall\, B_1 \in \mathcal{B} \ \ \forall\, B_2 \in \mathcal{B} \\ \forall\, x \in B_1 \cap B_2 \ \exists\, B_3 \in \mathcal{B} :\ (x \in B_3)\&(B_3 \subset B_1 \cap B_2)\big)\big\}\end{gathered} \tag{1.11}$$

is the family of all open bases of topologies on X. Under $\mathcal{B} \in (\mathrm{BAS})[X]$, we obtain that $\{\cup\}(\mathcal{B}) \in (\mathrm{top})[X]$. Then, for $\tau \in (\mathrm{top})[X]$

$$(\tau - \mathrm{BAS})_0[X] \triangleq \{\mathcal{B} \in (\mathrm{BAS})[X] |\, \tau = \{\cup\}(\mathcal{B})\}$$

1. General notions and designations

We use the standard set-theoretical symbolics (quatifiers, connectives and so on); $\emptyset$ is anempty set and $\overset{\triangle}{=}$ is the equality by definition. We call a set by a family in the case when every element of this set is a set also. We take the axiom of choice.

For every objects x and y, we denote by $\{x;y\}$ the set containing x and y and not containing no other elements. If h is an object, then we suppose that $\{h\} \overset{\triangle}{=} \{h;h\}$. Of course, sets are objects. Therefore, by [10, ch. II, §3], for every objects u and v, we suppose that $(u,v) \overset{\triangle}{=} \big\{\{u\};\{u;v\}\big\}$ receiving the ordered pair with first element u and second element v. If z is an arbitrary ordered pair, then by $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ we denote the first and second elements of z respectively; of course, $z = \big(\mathrm{pr}_1(z), \mathrm{pr}_2(z)\big)$ and $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ are defined uniquely.

If X is a set, then by $\mathcal{P}(X)$ we denote the family of all subsets of X and suppose that $\mathrm{Fin}(X)$ is the family of all finite nonempty subsets of X; of course. $\mathrm{Fin}(X) \subset \mathcal{P}'(X)$, where $\mathcal{P}'(X) \overset{\triangle}{=} \mathcal{P}(X)\backslash\{\emptyset\}$ is the family of all nonempty subsets of X. In addition, a family can be used as X. For every nonempty family $\mathfrak{X}$, we suppose that

$$\begin{gathered}\{\cup\}(\mathfrak{X}) \overset{\triangle}{=} \Big\{\bigcup_{X \in \mathcal{X}} X : \mathcal{X} \in \mathcal{P}(\mathfrak{X})\Big\}, \quad \{\cap\}(\mathfrak{X}) \overset{\triangle}{=} \Big\{\bigcap_{X \in \mathcal{X}} X : \mathcal{X} \in \mathcal{P}'(\mathfrak{X})\Big\}, \\ \{\cup\}_\sharp(\mathfrak{X}) \overset{\triangle}{=} \Big\{\bigcup_{X \in \mathcal{K}} X : \mathcal{K} \in \mathrm{Fin}(\mathfrak{X})\Big\}, \quad \{\cap\}_\sharp(\mathfrak{X}) \overset{\triangle}{=} \Big\{\bigcap_{X \in \mathcal{K}} X : \mathcal{K} \in \mathrm{Fin}(\mathfrak{X})\Big\};\end{gathered} \tag{1.1}$$

of course, every family of (1.1) is contained in $\mathcal{P}\big(\bigcup\limits_{X \in \mathfrak{X}} X\big)$ and contains $\mathfrak{X}$. For any set $\mathbb{M}$ and $\mathcal{M} \in \mathcal{P}'\big(\mathcal{P}(\mathbb{M})\big)$,

$$\mathbf{C}_{\mathbb{M}}[\mathcal{M}] \overset{\triangle}{=} \{\mathbb{M} \setminus M : \ M \in \mathcal{M}\} \in \mathcal{P}'\big(\mathcal{P}(\mathbb{M})\big). \tag{1.2}$$

In addition (see (1.2)), for any set S and a family $\mathcal{S} \in \mathcal{P}'\big(\mathcal{P}(S)\big)$, the equality $\mathcal{S} = \mathbf{C}_S\big[\mathbf{C}_S[\mathcal{S}]\big]$ is realized. If $\mathcal{A}$ is a nonempty family and B is a set, then

$$\mathcal{A}|_B \overset{\triangle}{=} \{A \cap B : \ A \in \mathcal{A}\} \in \mathcal{P}'\big(\mathcal{P}(B)\big) \tag{1.3}$$

is trace of $\mathcal{A}$ on the set B. Usually, in (1.3), the variant $\mathcal{A} \in \mathcal{P}'\big(\mathcal{P}(\mathbb{A})\big)$ and $B \in \mathcal{P}(\mathbb{A})$, where $\mathbb{A}$ is a set, is considered.

For any sets A and B, by B^A the set of all mappings from A into B is denoted. Under $f \in B^A$ and $a \in A$, by $f(a), f(a) \in B$, the value of f at the point a is denoted. For $f \in B^A$ and $C \in \mathcal{P}(A)$, we suppose that $f^1(C) \overset{\triangle}{=} \{f(x) : \ x \in C\}$; of course, $f^1(C) \subset B$ and

$$(C \neq \emptyset) \Rightarrow \big(f^1(C) \neq \emptyset\big).$$

Special families. In given item, we fix a set I (the case $I = \emptyset$ is not excluded). In the form of

$$\pi[I] \overset{\triangle}{=} \{\mathcal{I} \in \mathcal{P}'\big(\mathcal{P}(I)\big) \,|\, (\emptyset \in \mathcal{I})\&(I \in \mathcal{I})\&(A \cap B \in \mathcal{I} \ \ \forall A \in \mathcal{I} \ \ \forall B \in \mathcal{I})\}, \tag{1.4}$$

we have the family of all π-systems of subsets of I with «zero» and «unit». In terms of

$$(\mathrm{LAT})[I] \overset{\triangle}{=} \big\{\mathcal{L} \in \mathcal{P}'\big(\mathcal{P}(I)\big) \,|\, (\emptyset \in \mathcal{L})\&\big(\forall A \in \mathcal{L} \ \ \forall B \in \mathcal{L} \ \ (A \cup B \in \mathcal{L})\&(A \cap B \in \mathcal{L})\big)\big\} \tag{1.5}$$

(the family of all lattices of subsets of I), we define (see (1.4)) the basic family

$$(\mathrm{LAT})_0[I] \overset{\triangle}{=} \{\mathcal{I} \in (\mathrm{LAT})[I] \,|\, I \in \mathcal{I}\} = \{\mathcal{I} \in \pi[I] \,|\, A \cup B \in \mathcal{I} \ \ \forall A \in \mathcal{I} \ \ \forall B \in \mathcal{I}\} \tag{1.6}$$

SOME REPRESENTATIONS CONNECTED WITH ULTRAFILTERS AND MAXIMAL LINKED SYSTEMS

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Abstract: Ultrafilters and maximal linked systems (MLS) of a lattice of sets are considered. Two following variants of topological equipment are investigated: the Stone and Wallman topologies. These two variants are used both in the case of ultrafilters and for space of MLS. Under Wallman equipment, an analog of superextension is realized. Namely, the space of MLS with topology of the Wallman type is supercompact topological space. By two above-mentioned equipments a bitopological space is realized.

Key words: Lattice, Linked system, Ultrafilter.

Introduction

In connection with the supercompactness property, maximal linked systems (MLS) of closed sets in a topological space (TS) are investigated (see [1–4]; in particular, we note the important statement of [3] about supercompactness of metrizable compactums). The space of «closed» MLS with topology of the Wallman type is a superextension of the initial TS.

Now, following [5], we consider more general approach. Namely, we suppose that a lattice of subsets of arbitrary nonempty set is given. Then, MLS of sets of this lattice are investigated. In particular, the lattice of closed sets in a TS can be used. Then, we obtain the above-mentioned variant of [1–4]. But, many other realizations are possible. For example, we can consider an algebra of sets as variant of the above-mentioned lattice. Note by the way, that in this case the Stone topology on the ultrafilter space is very natural. Since in many respects, MLS are similar to ultrafilters, the Stone equipment is submitted natural and for space of MLS. So, the idea of emploument of the two types of topologies arises: we keep in mind the Wallman and Stone variants.

We recall that ultrafilters were used as generalized elements in problems connected with attainability under constraints of asymptotic character (see, for example, [6–8]). Now, we seek to explore spaces which are comprehending for ultrafilters. In this article, it is established that the space of MLS is comprehending in this sense. In addition, it is logical to consider two characteristic types of topologies both for ultrafilters and for MLS. And what is more, we obtain two bitopological spaces (as a bitopological space, we consider every set equipped with two comparable topologies; in this connection, we note monograph [9]).

The case when two above-mentioned topologies coincide, we consider as degenerate. In the following, characteristic cases of such degeneracy are established (a variant of non-degenerate realization of bitopological space specified also). We indicate important types of lattices for which above-mentioned constructions are realized sufficiently understandably.

8. **Gashkov S.B.** Bernstein's inequality, Riesz's identity and Euler's formula for the sum of reciprocal squares // Mat. Pros., Ser. 3, 2014. Vol. 18. P. 143–171.
9. **Gorin E.A.** Bernstein inequalities from the operator theory point of view // Vestnik Kharkov. Univ. Prikl. Mat. Mekh.,1980. Vol. 45. P. 77–105.
10. **Kahane J.-P.** Séries de Fourier absolument convergentes. Berlin: Springer, 1970.
11. **Kozko A.I.** The exact constants in the Bernstein–Zygmund–Szegő inequalities with fractional derivatives and the Jackson–Nikolskii inequality for trigonometric polynomials // East J. Approx., 1998. Vol. 4, no. 3. P. 391–416.
12. **Lieb E., Loss M.** Analysis. Graduete Studies in Math., vol. 14. AMS, Providence, Rhode Island, 1997.
13. **Lizorkin P.I.** Bounds for trigonometrical integrals and an inequality of Bernstein for fractional derivatives // Izv. Akad. Nauk SSSR Ser. Mat., 1965. Vol. 29, no. 1. P. 109–126.
14. **Lukacs E.** Characteristic functions. London: Griffin, 1970.
15. **Manov A., Zastavnyi V.** Positive definiteness of piecewise-linear function // Expositiones Mathematicae, 2017. Vol. 35. P. 357–361. DOI: 10.1016/j.exmath.2016.12.002
16. **Riesz M.** Formule d'interpolation pour la dérivée d'un polynome trigonométrique // C. R. Acad. Sci. 1914. Vol. 158. P. 1152–1154.
17. **Riesz M.** Eine trigonometrische interpolationsformel und einige ungleichungen für polynome // Jahresbericht der Deutschen Mathematiker-Vereinigung, 1914. Vol. 23. P. 354–368.
18. **Rudin W.** Real and complex analysis. Singapore: McGraw-Hill Int. Editions, 3nd Ed., 1987.
19. **Sasvári Z.** Positive definite and definitizable functions. Berlin: Akademie Verlag, 1994.
20. **Szegő G.** Über einen Satz des Herrn Serge Bernstein // Schriften der Königsberger Gelehrten Gesellschaft. 1928. Vol. 5, no. 4. P. 59–70.
21. **Sz.-Nagy B.** Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen. I. Periodischer Fall // Berichte der Sächsischen Akademie der Wissenschaften zu Leipzig, 1938. Vol. 90. P. 103–134.
22. **Timan A.F.** Theory of approximation of functions of a real variable. Oxford: Pergamon Press, 1963.
23. **Trigub R.M., Belinsky E.S.** Fourier analysis and approximation of functions. Dordrecht: Kluwer Acad. Publ., 2004.
24. **Vakhaniya N.N., Tarieladze V.I., and Chobanyan S.A.** Probability distributions in Banach spaces. M.: Nauka, 1985. [in Russian]
25. **Vinogradov O.L.** Sharp error estimates for the numerical differentiation formulas on the classes of entire functions of exponential type // Siberian Math. J., 2007. Vol. 48, no. 3. P. 430–445. DOI: 10.1007/s11202-007-0046-9
26. **Williamson R.E.** Multiply monotone functions and their Laplace transforms // Duke Math. J., 1956. Vol. 23, no. 2. P. 189–207. DOI: 10.1215/S0012-7094-56-02317-1
27. **Zastavnyi V.P.** On positive definiteness of some functions // Journal of Multivariate Analysis, 2000. Vol. 73, no. 1. P. 55–81. DOI: 10.1006/jmva.1999.1864
28. **Zygmund A.** Trigonometric series. Vol. II. Cambridge: Cambridge Univ. Press, 1959.

Remark 9. It is not difficult to see that all the arguments in the proof of Theorem 9 remain in force also in the case where the 2-periodic continuous function ψ is expanded in an absolutely convergent Fourier series (without the assumption of nonnegativity of the Fourier coefficients $c_k(\psi)$). Therefore, the following statement holds: *Assume that a* 2*-periodic function* $\psi \in C(\mathbb{R})$ *is expanded into an absolutely convergent Fourier series and* $\beta \in \mathbb{R}$*. Then equality* (8.1) *holds for any function* $f \in C(\mathbb{T})$ *such that the series on the left in* (8.1) *converges uniformly on* $\mathbb{T}$*.*

9. Conclusion

In conclusion, we point out some problems which, in our opinion, have not been solved yet.

1) To prove or disprove that only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal in the Berstein–Szegő inequality (6.2) for $r = 1$ and $\beta = 0$ (the case of the derivative of the adjoint polynomial) when $p = \infty$ or $p = 1$. When $p = \infty$, this case was distinguished in the paper by Szegő [20, p. 66]. We note that the arguments in the monographs by Zygmund [28, Ch. X, Sect. 3, (3.24)] and Akhiezer [1, Sect. 84, p. 189] corresponding to this case are not correct, since some coefficients in the interpolation formulas are zero (see [28, Ch. X, Sect. 3, (3.22)] for $\alpha = \pi/2$ and [1, Sect. 84, p. 188, (II)] for $\alpha = 0$).

2) Let $n \in \mathbb{N}$, and let, for a trigonometric polynomial $f \in \mathscr{F}_n$, condition (5.9) or (5.10) be satisfied for all integers $s = 0, \ldots, 2n - 1$. Then $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$ (see Remark 4). The question is, which values of s can be left to have the same conclusion? This is a more general problem than the previous one.

3) To prove or disprove that if, for some $s \in \mathbb{Z}$, inequalities $\mu_s(n, \psi) > 0$ and $\mu_{s+1}(n, \psi) > 0$ hold and a function $f \in C(\mathbb{T})$ is extremal in inequality (5.8) with $p = 1$, then $f(t) = h(t)g(t)$, where the function h belongs to $L_\infty(\mathbb{T})$ and has the form (5.4), $g \in C(\mathbb{T})$, and $g(t) \geq 0$ for $t \in \mathbb{R}$. This is true if, in addition, $f(t) \neq 0$ for almost all $t \in \mathbb{R}$ with respect to the Lebesgue measure (see Remark 4 for the case $p = 1$).

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REFERENCES

1. **Akhiezer N.I.** Lectures on approximation theory. Moscow: Nauka, 1965. [in Russian]
2. **Akhiezer N.I.** Lectures on integral transforms. Kharkov: Vishcha Shkola, 1984. [in Russian]
3. **Arestov V.V.** On integral inequalities for trigonometric polynomials and their derivatives // Math. USSR-Izv., 1982. Vol. 18, no. 1. P. 1–17. DOI: 10.1070/IM1982v018n01ABEH001375
4. **Arestov V.V.** Sharp inequalities for trigonometric polynomials with respect to integral functionals // Proc. Steklov Inst. Math., 2011. Vol. 273, suppl. 1. P. 21–36. DOI: 10.1134/S0081543811050038
5. **Arestov V.V., Glazyrina P.Yu.** Bernstein–Szegő inequality for fractional derivatives of trigonometric polynomials // Proc. Steklov Inst. Math., 2015. Vol. 288, suppl. 1. P. 13–28. DOI: 10.1134/S0081543815020030
6. **Boas R.P., Jr.** Entire functions. New York: Academic Press, 1954.
7. **Bisgaard T.M., Sasvári Z.** Characteristic functions and moment sequences: positive definiteness in probability. New York: Nova Sci. Publishers, 2000.

P r o o f. Consider operator (1.1) for the function $\varphi(x) \equiv e^{i\beta x}\psi(x)$. Under the conditions of the theorem, we can put the sign of equality in relation (3.2) for $\varepsilon = 1/n$ and $\tau = 1$. Therefore, the left-hand side of equality (5.6) can be replaced by the sum of the series in (3.2). We obtain identity (8.1) with accuracy up to the factor $e^{-i\beta}$. The specified properties of the numbers $\mu_k(n,\psi)$ follow from (5.5) and (5.7). $\square$

Corollary 3. *Assume that $g \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\operatorname{supp} g \subset [-1,1]$, $\beta \in \mathbb{R}$, and $n \in \mathbb{N}$. Then, for any polynomial $f \in \mathscr{F}_{2n}$, the following equality holds*:

$$\begin{aligned}&\sum_{|k|\le 2n}\left(\operatorname{Re} g\left(1-\frac{|k|}{n}\right)+i\operatorname{sign}k\operatorname{Im} g\left(1-\frac{|k|}{n}\right)\right)e^{i\beta\operatorname{sign}k}c_k(f)e^{ikt}\\&=\sum_{k=0}^{2n-1}(-1)^k f\left(t+\frac{\pi k+\beta}{n}\right)\mu_k(n,g,\beta),\quad t\in\mathbb{R},\end{aligned}\tag{8.2}$$

where $\mu_k(n,g,\beta) = \sum_{m\in\mathbb{Z}} \widehat{g}(-\beta-(k+2nm)\pi)/2$, $k \in \mathbb{Z}$, and $\sum_{k=0}^{2n-1} \mu_k(n,g,\beta) = g(0)$.

P r o o f. Let ψ be a 2-periodic function, and let $\psi(x) = g(-x)e^{-i\beta x}$ for $x \in [-1,1]$. Then $\psi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and

$$\psi(x-1) = e^{-i\beta x}e^{i\beta\operatorname{sign}x}(\operatorname{Re} g(1-|x|)+i\operatorname{sign}x\operatorname{Im} g(1-|x|)),\quad |x|\le 2.$$

It remains to take into account that $c_k(\psi) = \widehat{g}(-\beta-k\pi)/2$, $k \in \mathbb{Z}$. $\square$

Remark 8. We note that if, for g, we take the function $g_r(x) = (1-|x|)_+^r$, $r \ge 1$, then, in (8.2), we obtain the interpolation formula of A.I. Kozko [11] (and of M. Riesz and of G. Szegő for $r = 1$) for the Weyl–Nagy derivative:

$$f^{(r,\beta)}(t) = n^r\sum_{k=0}^{2n-1}(-1)^k f\left(t+\frac{\pi k+\beta}{n}\right)\mu_k(n,g_r,\beta),\quad t\in\mathbb{R},\quad f\in\mathscr{F}_n;\quad \sum_{k=0}^{2n-1}\mu_k(n,g_r,\beta)=1,$$

where $\mu_k(n,g_r,\beta) > 0$ for all $n \in \mathbb{N}$, $k = 0,\dots,2n-1$, $\beta \in \mathbb{R}$, and $r > 1$. These coefficients are also positive for $r = 1$ if $n = 1$ and $\beta \in \mathbb{R}$ or if $n \ge 2$ and $\beta \ne q\pi$, $q \in \mathbb{Z}$. If $r = 1$, $n \ge 2$, and $\beta = \pi q$ with $q \in \mathbb{Z}$, then, the number of positive coefficients among $\mu_k(n,g_1,\beta)$, $k = 0,\dots,2n-1$, is $n+1$ and the remaining are zero (see Remark 7). For $r = 1$, these coefficients are easily calculated. Since $\widehat{g_1}(t) = 2(1-\cos t)/t^2$, we have

$$\mu_k(n,g_1,\beta)=\frac{1-(-1)^k\cos\beta}{4n^2}\sum_{m\in\mathbb{Z}}\frac{1}{\left(\dfrac{\beta+k\pi}{2n}+m\pi\right)^2}=\frac{1-(-1)^k\cos\beta}{2n^2\left(1-\cos\dfrac{\beta+k\pi}{n}\right)}>0,\quad \beta\ne q\pi,\quad q\in\mathbb{Z},$$

For $\beta = q\pi$ with $q \in \mathbb{Z}$, we can restrict ourselves to the case $\beta = 0$ (see Remark 6): $\mu_{2k}(n,g_1,0) = 0$ for $k = 1,\dots,n-1$ (if $n \ge 2$), $\mu_0(n,g_1,0) = 1/2$, and

$$\mu_{2k-1}(n,g_1,0)=\frac{1}{n^2\left(1-\cos\dfrac{(2k-1)\pi}{n}\right)}>0,\quad k=1,\dots,n.$$

Let a function $g \in C(\mathbb{R})$ be even, nonnegative, decreasing, and convex on $(0, +\infty)$, and let $\operatorname{supp} g \subset [-1, 1]$. Assume that $n \in \mathbb{N}$, $n \geq 2$, and $-1/n \leq h \leq 1 - 1/n$. Let $g_{1/n,h}(x) := hg(x) + (1 - 1/n - h)g(nx)$, $x \in \mathbb{R}$. It follows from Corollary 2 that $g_{1/n,h} \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$. Since $\operatorname{supp} g \subset [-1, 1]$, we have $\operatorname{supp} g_{1/n,h} \subset [-1, 1]$. Therefore, for the function $g_{1/n,h}$, we can construct operator (6.4) which does not depend on the parameter θ, since $\operatorname{Im}(g_{1/n,h}) \equiv 0$. It is not difficult to verify that, for polynomials $f \in \mathscr{F}_{2n}$, the following equality holds:

$$D_{n,0}^{g_{1/n,h},\beta}(f)(t) = hD_{n,0}^{g,\beta}(f)(t) + (1 - 1/n - h)g(0)R_n^\beta(f)(t), \tag{7.1}$$

where

$$R_n^\beta(f)(t) := e^{-i\beta} \sum_{|k|=n} e^{i\beta \operatorname{sign} k} c_k(f) e^{ikt} = \frac{e^{-i\beta}}{\pi} \int_{-\pi}^{\pi} \cos(nx - \beta) f(t + x)\, dx. \tag{7.2}$$

We note that $g_{1/n,h}(0) = (1 - 1/n)g(0)$. In addition, if $g(x) = (1 - |x|)_+^r$, $r \geq 1$, then $D_{n,\theta}^{g_r,\beta}(f)(t) \equiv e^{-i\beta} f^{(r,\beta)}(t)/n^r$ for any polynomial $f \in \mathscr{F}_n$. We write Theorem 6 for the operator (7.1) and restrict ourselves only to inequality (6.6).

Theorem 8. *Let a function $g \in C(\mathbb{R})$ be even, nonnegative, decreasing, and convex on $(0, +\infty)$, and let $\operatorname{supp} g \subset [-1, 1]$. Assume that $n \geqslant 2$, $-1/n \leq h \leq 1 - 1/n$, $\beta \in \mathbb{R}$, and $1 \leqslant p \leqslant \infty$. Then, for any polynomial $f \in \mathscr{F}_{2n}$, we have*

$$\left\| hD_{n,0}^{g,\beta}(f) + (1 - 1/n - h)g(0)R_n^\beta(f) \right\|_p \leqslant (1 - 1/n)g(0)\|f\|_p. \tag{7.3}$$

If $r \geq 1$, then, for any polynomial $f \in \mathscr{F}_n$, we have

$$\left\| hf^{(r,\beta)}/n^r + (1 - 1/n - h)e^{i\beta} R_n^\beta(f) \right\|_p \leqslant (1 - 1/n)\|f\|_p. \tag{7.4}$$

Inequalities (7.3) *and* (7.4) *turn into equalities for polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$.*

Without proof, we note that if the function g in Theorem 8 is not piecewise linear on $[0, +\infty)$ with equidistant nodes, then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal in inequality (7.3) with $p \in (1, \infty)$. When $p = 1$ or $p = \infty$, a similar conclusion holds, but for the class of trigonometric polynomials of degree at most n. If $r > 1$, then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal in inequality (7.4).

8. Interpolation formulas for periodic functions

If the trigonometric series on the right-hand side of relation (3.2) converges uniformly on $\mathbb{T}$, then one can put the sign of equality in this relation and the obtained equality can be regarded as some interpolation formula. We explain this with the example of the following theorem.

Theorem 9. *Assume that $n \in \mathbb{N}$, a 2-periodic function ψ belongs to $\Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\beta \in \mathbb{R}$, and the numbers $\mu_k(n, \psi)$ are defined by formula* (5.5). *Then the identity*

$$\sum_{k \in \mathbb{Z}} e^{i\beta k/n} \psi\left(\frac{k}{n} - 1\right) c_k(f) e^{ikt} \equiv \sum_{k=0}^{2n-1} (-1)^k f\left(t + \frac{\pi k + \beta}{n}\right) \mu_k(n, \psi) \tag{8.1}$$

holds for any function $f \in C(\mathbb{T})$ such that the series on the left converges uniformly on $\mathbb{T}$. Moreover, $\mu_0(n, \psi) + \ldots + \mu_{2n-1}(n, \psi) = \psi(0)$, $c_k(\psi) \geq 0$, $k \in \mathbb{Z}$, $\mu_k(n, \psi) \geq 0$, $k = 0, \ldots, 2n - 1$, and $\mu_k(n, \psi) = 0$ for some $k = 0, \ldots, 2n - 1$ if and only if $c_{k+2nm}(\psi) = 0$ for all $m \in \mathbb{Z}$.

among $\mu_s(n, g_1, 0)$, $s = 0, \dots, 2n-1$, is $n+1$ and the remaining are zero. The latter property also holds for any $\beta = \pi q$ with $q \in \mathbb{Z}$ (see Remark 6).

Thus, only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal in inequalities (6.1) and (6.2) under conditions (A_1) and (B_1) or (A_2) and (B_2), where

(A_1) $r > 1$ and $\beta \in \mathbb{R}$; or $r = 1$, $\beta \in \mathbb{R}$, and $n = 1$; or $r = 1$, $\beta \neq \pi q$, $q \in \mathbb{Z}$, and $n \geq 2$;

(B_1) the function J is strictly increasing on $(0, +\infty)$ for (6.1) or $1 \leq p \leq \infty$ for (6.2);

(A_2) $r = 1$, $\beta = \pi q$ with $q \in \mathbb{Z}$, and $n \geq 2$;

(B_2) the function J is strictly convex on $(0, +\infty)$ for (6.1) or $1 < p < \infty$ for (6.2).

The case where $r = 1$, $\beta = \pi q$ with $q \in \mathbb{Z}$, $n \geq 2$, and $p = 1$ or $p = \infty$ has not been studied.

7. Case of piecewise linear functions

In [15], the following R.M. Trigub problem on the positive definiteness of piecewise linear functions was solved. For given $\alpha \in (0, 1)$ and $h \in \mathbb{R}$, the function $f_{\alpha,h} : \mathbb{R} \to \mathbb{C}$ is defined as follows: (1) the function $f_{\alpha,h}$ is even; (2) $f_{\alpha,h}(x) = 0$ for $x > 1$, the function $f_{\alpha,h}$ is linear on each of the intervals $[0, \alpha]$ and $[\alpha, 1]$, $f_{\alpha,h}(0) = 1$, $f_{\alpha,h}(\alpha) = h$, and $f_{\alpha,h}(1) = 0$. For any fixed $\alpha \in (0, 1)$, it is required to find the set of all $h \in \mathbb{R}$ such that the piecewise linear function $f_{\alpha,h}$ is positive definite on $\mathbb{R}$. If $0 \leq h \leq 1 - \alpha$, then the continuous even function $f_{\alpha,h}(x)$ is convex on $(0, +\infty)$, $f_{\alpha,h}(+\infty) = 0$, and, hence, it is positive definite by the Pólya theorem (see, for instance, [14, Theorem 4.3.1]). A complete description of such $h \in \mathbb{R}$ is given in the following theorem.

Theorem 7 [15]. *Let $\alpha \in (0, 1)$ and $h \in \mathbb{R}$. Then $f_{\alpha,h} \in \Phi(\mathbb{R})$ if and only if $m(\alpha) \leq h \leq 1 - \alpha$, where $m(\alpha) = 0$ if $1/\alpha \notin \mathbb{N}$ and $m(\alpha) = -\alpha$ if $1/\alpha \in \mathbb{N}$.*

From Theorem 7, we obtain the following sufficient condition for the positive definiteness.

Corollary 2. *If a function $g \in C(\mathbb{R})$ is even, nonnegative, decreasing, and convex on $(0, +\infty)$, then, for $\alpha \in (0, 1)$, $1/\alpha \in \mathbb{N}$, and $-\alpha \leq h \leq 1 - \alpha$, the function $g_{\alpha,h}(x) := hg(x) + (1 - \alpha - h)g(x/\alpha)$ belongs to the class $\Phi(\mathbb{R})$.*

The nontrivial case here is when $-\alpha \leq h < 0$.

P r o o f. The function g is represented in the form (see, for instance, [26])

$$g(x) = \int_0^{+\infty} (1 - |sx|)_+ \, d\mu(s), \quad x \in \mathbb{R},$$

where μ is a nonnegative finite Borel measure on $[0, +\infty)$. Obviously,

$$g_{\alpha,h}(x) = (1 - \alpha) \int_0^{+\infty} f_{\alpha,h}(sx) \, d\mu(s), \quad x \in \mathbb{R}.$$

For the specified α and h, we have $f_{\alpha,h} \in \Phi(\mathbb{R})$. Hence, $g_{\alpha,h} \in \Phi(\mathbb{R})$ (see, for instance, [27, Lemma 1]). $\square$

One can use the positive definite function $g_{\alpha,h}$ given in Corollary 2 to obtain new sharp inequalities for trigonometric polynomials.

functions $(-1)^s f(t+\pi s/n)$ *are identical on* $\mathbb{R}$ *for all* $s=0,\dots,2n-1$ *such that* $\mu_s(n,g,\beta,\theta)>0$, *where, for* $k\in\mathbb{Z}$,

$$4\mu_k(n,g,\beta,\theta)=(1+\theta)\sum_{m\in\mathbb{Z}}\widehat{g}(-\beta-(k+2nm)\pi)+(1-\theta)\sum_{m\in\mathbb{Z}}\widehat{g}(\beta+(k+2nm)\pi). \tag{6.7}$$

If, in addition, for some $s\in\mathbb{Z}$, *the inequalities* $\mu_s(n,g,\beta,\theta)>0$ *and* $\mu_{s+1}(n,g,\beta,\theta)>0$ *hold, then inequality* (6.5) *or inequality* (6.6) *with* $p\in(1,\infty)$ *turns into an equality only at the polynomials* $f(t)=ce^{int}+\nu e^{-int}$, $c,\nu\in\mathbb{C}$.

3) *When* $p=\infty$, *inequality* (6.6) *turns into an equality at some polynomial* $f\in\mathscr{F}_{2n}$ *if and only if, for some* $\eta,\delta\in\mathbb{R}$, *the equality* $(-1)^s f(\eta+\pi s/n)=e^{i\delta}\|f\|_\infty$ *holds for all* $s=0,\dots,2n-1$ *such that* $\mu_s(n,g,\beta,\theta)>0$.

If $\mu_s(n,g,\beta,\theta)>0$ *for* $s=0,\dots,2n-1$, *then only polynomials of the form* $f(t)=ce^{int}+\nu e^{-int}$, $c,\nu\in\mathbb{C}$, *are extremal among trigonometric polynomials of degree at most* n *for which inequality* (6.6) *with* $p=\infty$ *turns into an equality.*

4) *When* $p=1$, *inequality* (6.6) *turns into an equality at some polynomial* $f\in\mathscr{F}_{2n}$ *if and only if, for any* $t\in\mathbb{R}$, *there exists a number* $\delta(t)\in\mathbb{R}$ *such that the identity* $(-1)^s f(t+\pi s/n)\equiv e^{i\delta(t)}\,|f(t+\pi s/n)|$ *holds for all* $s=0,\dots,2n-1$ *such that* $\mu_s(n,g,\beta,\theta)>0$.

If a polynomial $f\in\mathscr{F}_q$, $1\le q<2n$, *is extremal in inequality* (6.6) *with* $p=1$, *then any polynomial of the form* $cf(t)g(t)$, *where* $c\in\mathbb{C}$, $g\in\mathscr{F}_{2n-q}$, *and* $g(t)\ge 0$ *for all* $t\in\mathbb{R}$, *is also extremal. In particular, polynomials of the form* $(ce^{int}+\nu e^{-int})g(t)$, *where* $c,\nu\in\mathbb{C}$ *and* g *is an arbitrary nonnegative trigonometric polynomial of degree at most* n, *are extremal in inequality* (6.6) *with* $p=1$.

5) *If* $\mu_s(n,g,\beta,\theta)>0$ *for all* $s=0,\dots,2n-1$ *and the function* J *is strictly increasing on* $(0,+\infty)$, *then only polynomials of the form* $f(t)=ce^{int}+\nu e^{-int}$, $c,\nu\in\mathbb{C}$, *are extremal among trigonometric polynomials of degree at most* n *for which inequality* (6.5) *or inequality* (6.6) *with* $p=1$ *turns into an equality.*

Remark 6. If $q\in\mathbb{Z}$ and $q=2nl+r$, where $l,r\in\mathbb{Z}$ and $0\le r\le 2n-1$, then

$$\mu_k(n,g,\beta+\pi q,\theta)=\begin{cases}\mu_{k+r}(n,g,\beta,\theta), & 0\le k\le 2n-1-r,\\ \mu_{k+r-2n}(n,g,\beta,\theta), & 2n-r\le k\le 2n-1,\quad r\ge 1.\end{cases}$$

Remark 7. Inequalities (6.1) and (6.2) follow from inequalities (6.5) and (6.6) if, for g, we take the function $g_r(x)=(1-|x|)_+^r$ which is positive definite for $r\ge 1$ (the Pólya property). Since $g_r(1-|x|)=|x|^r$ for $|x|\le 1$, we have $D_{n,\theta}^{g_r,\beta}(f)(t)\equiv e^{-i\beta}f^{(r,\beta)}(t)/n^r$ for any polynomial $f\in\mathscr{F}_n$, $n\in\mathbb{N}$. In our case, the values (6.7) are independent of θ and such that

$$\mu_k(n,g_r,\beta)=\sum_{m\in\mathbb{Z}}\widehat{g_r}(\beta+(k+2nm)\pi))/2,\quad k\in\mathbb{Z}\,.$$

It is well known that, for $r>1$, the Fourier transform $\widehat{g_r}(t)$ is positive for all $t\in\mathbb{R}$ (see, for instance, [27, Lemma 7, $n=\lambda=\delta=1$]). Therefore, $\mu_s(n,g_r,\beta)>0$ for all $r>1$, $\beta\in\mathbb{R}$, $n\in\mathbb{N}$, and $s\in\mathbb{Z}$.

For $r=1$, the Fourier transform of the function g_1 is easily calculated and is equal to $\widehat{g_1}(t)=2(1-\cos t)/t^2$. Obviously, $\widehat{g_1}(t)=0$ only for $t=2q\pi$ with $q\in\mathbb{Z}$, $q\neq 0$. Therefore, if $\beta\neq q\pi$, $q\in\mathbb{Z}$, then $\mu_s(n,g_1,\beta)>0$ for all $n\in\mathbb{N}$ and $s\in\mathbb{Z}$.

If $\beta=0$ and $n\in\mathbb{N}$, then: (1) $\mu_s(n,g_1,0)>0$ for $s=0$ and for all odd $s\in[1,2n-1]$; (2) $\mu_s(n,g_1,0)=0$ for all even $s\in[2,2n-1]$ if $n\ge 2$. In this case, the number of positive values

Then

$$\psi(x-1) = e^{-i\beta x} \begin{cases} g(1-|x|)e^{i\beta}, & 0 \le x \le 2; \\ g(|x|-1)e^{-i\beta}, & -2 \le x \le 0. \end{cases}$$

Taking into account that the real and imaginary parts of a positive definite function are even and odd functions, respectively, we obtain the equality $\psi(x-1) = e^{-i\beta x} e^{i\beta \operatorname{sign} x} g_0(x)$, $|x| \le 2$, where

$$g_0(x) = \operatorname{Re} g(1-|x|) + i \operatorname{sign} x \operatorname{Im} g(1-|x|), \quad |x| \le 2.$$

Obviously, the function $\varphi(x) := e^{i\beta x}\psi(x)$ belongs to $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ and

$$\varphi(x-1) = e^{-i\beta} g_0(x) e^{i\beta \operatorname{sign} x} = e^{-i\beta}(\operatorname{Re} g(1-|x|) + i \operatorname{sign} x \operatorname{Im} g(1-|x|)) e^{i\beta \operatorname{sign} x}, \quad |x| \le 2. \tag{6.3}$$

Consider the operator $A_{1/n,1}$ generated by the function φ by formula (1.1) for $\varepsilon = 1/n$ and $\tau = 1$. We can apply Theorem 5 and Remarks 4 and 5 to this operator. It should be taken into account that $\psi(0) = g(0)$ and $c_k(\psi) = \widehat{g}(-\beta - k\pi)/2$, $k \in \mathbb{Z}$. For polynomials $f \in \mathscr{F}_{2n}$, the operator $A_{1/n,1}$ has the following form (see (3.2) and (6.3)):

$$A_{1/n,1}(f)(t) \equiv e^{-i\beta} \sum_{|k| \le 2n} \left(\operatorname{Re} g\left(1 - \frac{|k|}{n}\right) + i \operatorname{sign} k \operatorname{Im} g\left(1 - \frac{|k|}{n}\right)\right) e^{i\beta \operatorname{sign} k} c_k(f) e^{ikt}.$$

We introduce one more parameter. Obviously, for any $\theta \in [-1,1]$, the function

$$g_\theta(x) := ((1+\theta)g(x) + (1-\theta)g(-x))/2 = \operatorname{Re} g(x) + i\theta \operatorname{Im} g(x), \quad x \in \mathbb{R},$$

also belongs to the class $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\operatorname{supp} g_\theta \subset [-1,1]$. Therefore, all the above arguments are applicable to the function g_θ as well. It should be taken into account that, for the corresponding function ψ_θ, we have $\psi_\theta(0) = g_\theta(0) = g(0)$ and

$$c_k(\psi_\theta) = ((1+\theta)\widehat{g}(-\beta - k\pi) + (1-\theta)\widehat{g}(\beta + k\pi))/4, \quad k \in \mathbb{Z}.$$

For the function $\varphi_\theta(x) := e^{i\beta x}\psi_\theta(x) \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, we consider the corresponding operator $A_{1/n,1}$ with $\varepsilon = 1/n$ and $\tau = 1$ (see (1.1)). We state the results obtained in Theorem 5 and Remarks 4 and 5 for the following operator defined on polynomials $f \in \mathscr{F}_{2n}$:

$$\begin{aligned} &D^{g,\beta}_{n,\theta}(f)(t) := A_{1/n,1}(f)(t) \equiv \\ &e^{-i\beta} \sum_{|k| \le 2n} \left(\operatorname{Re} g\left(1 - \frac{|k|}{n}\right) + i\theta \operatorname{sign} k \operatorname{Im} g\left(1 - \frac{|k|}{n}\right)\right) e^{i\beta \operatorname{sign} k} c_k(f) e^{ikt}. \end{aligned} \tag{6.4}$$

Theorem 6. *Assume that $g \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\operatorname{supp} g \subset [-1,1]$, $g(0) > 0$, $\beta \in \mathbb{R}$, $\theta \in [-1,1]$, and $1 \le p \le \infty$. Let J be a convex nondecreasing function on $[0,+\infty)$. Then:*

1) *For any $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{T}} J\left(|D^{g,\beta}_{n,\theta}(f)(t)|\right) dt \le \int_{\mathbb{T}} J\left(g(0)|f(t)|\right) dt, \quad f \in \mathscr{F}_{2n}, \tag{6.5}$$

$$\|D^{g,\beta}_{n,\theta}(f)\|_p \le g(0)\|f\|_p, \quad f \in \mathscr{F}_{2n}. \tag{6.6}$$

Inequalities (6.5) and (6.6) turn into equalities, for instance, for polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$.

2) *If the function J is strictly convex at any point of the interval $(0,+\infty)$, then inequality (6.5) or inequality (6.6) with $p \in (1,\infty)$ turns into an equality at some polynomial $f \in \mathscr{F}_{2n}$ if and only if the*

of degree at most n with coefficients in $\mathbb{C}$, where $a_k := c_k + c_{-k}$ and $b_k := i(c_k - c_{-k})$, $k \geq 0$. There are several different definitions of fractional derivative. The following operator for $r > 0$ and $\beta \in \mathbb{R}$ presumably first appeared in the paper by Sz.-Nagy [21, equality (2) for $m = 1$, $\lambda(k) = k^r$]. For $f \in \mathscr{F}_n$, we define

$$f^{(r,\beta)}(t) := \sum_{|k|\leq n} |k|^r e^{i\beta \operatorname{sign} k} c_k e^{ikt} = \sum_{k=1}^{n} k^r \left(a_k \cos\left(kt+\beta\right) + b_k \sin\left(kt+\beta\right)\right).$$

For $\beta = r\pi/2$, we obtain the Weyl derivative which, for $r \in \mathbb{N}$, coincides with the usual derivative of order r. Often, this operator is called the Weyl–Nagy derivative.

Let J be a convex and nondecreasing function on $[0, +\infty)$. Kozko proved (see [11, Theorem 1, Corollary 1]) that if $1 \leq p \leq \infty$, then, for any $n \in \mathbb{N}$, $r \geq 1$, and $\beta \in \mathbb{R}$, the following inequalities hold:

$$\int_{\mathbb{T}} J\left(\left|f^{(r,\beta)}(f)(t)\right|\right) dt \leq \int_{\mathbb{T}} J\left(n^r |f(t)|\right) dt, \quad f \in \mathscr{F}_n, \tag{6.1}$$

$$\|f^{(r,\beta)}\|_p \leq n^r \|f\|_p, \quad f \in \mathscr{F}_n. \tag{6.2}$$

For the usual derivative, i.e., when $r = 1$ and $\beta = \pi/2$, inequality (6.2) was proved by Bernstein in the case $p = \infty$. For $r = 1$ and $\beta \in \mathbb{R}$, inequality (6.2) was obtained by Szegő [20] in the case $p = \infty$ and inequality (6.1) was proved by Zygmund [28, Ch. X, Sect. 3, (3.25)] (his proof for real polynomials is also true for polynomials in $\mathscr{F}_n$). This and the identity

$$f^{(r+1,\beta)}(t) \equiv \left(f^{(r,\beta)}(t)\right)^{(1,0)}, \quad r > 0, \quad \beta \in \mathbb{R},$$

imply the validity of inequality (6.2) for any $r \in \mathbb{N}$. Inequality (6.2) for $p = \infty$, $r \geq 1$, $\beta = -r\pi/2$, and $\beta = 0$ (the case of the Riesz derivative) was proved by Lizorkin [13, Theorems 2, 2′].

Obviously, inequalities (6.1) and (6.2) turn into equalities for the polynomials $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$. Szegő [20, p. 66] proved that, in inequality (6.2) with $p = \infty$, there are no other extremal polynomials in the case $r = 1$ and $\beta \neq q\pi$, $q \in \mathbb{Z}$ (see also arguments in [1, Sect. 84, p. 189]). If, in addition, the function $tJ'(t)$ is strictly increasing on $(0, +\infty)$, then, in inequalities (6.1) and (6.2) for $1 \leq p < \infty$, $n \in \mathbb{N}$, $r \geq 1$, and $\beta \in \mathbb{R}$, there are no other extremal polynomials at least in the following cases (see [3, Corollary 6], [5, Theorems 1,2]): (1) in the case of the usual derivative of order $r \in \mathbb{N}$; (2) $n = 1$, $r \geq 1$, and $\beta \in \mathbb{R}$ or $n \geq 2$, $r \geq \ln(2n)/\ln(n/(n-1))$, and $\beta \in \mathbb{R}$.

For $r = 1$ and $\beta \neq q\pi$, $q \in \mathbb{Z}$, in inequalities (6.2) and (6.1) (if, in addition, the function $J(t)$ is strictly increasing on $(0, +\infty)$), only polynomials of the form $f(t) = a\cos nt + b\sin nt$, $a, b \in \mathbb{R}$, are extremal in the class of real trigonometric polynomials. This result is due to Zygmund [28, Ch. X, Sect 3, (3.24), (3.25)].

Other cases in which inequality (6.2) holds, when $r < 1$ or $0 \leq p < 1$, were considered in the paper by Arestov and Glazyrina [5], where these inequalities are called Bernstein–Szegő inequalities and a complete history of such inequalities is given.

Inequalities more general than (6.1) and (6.2) are obtained from Theorem 5 under an appropriate choice of the function ψ. The method of construction of the function ψ described below is essentially contained in the paper by Lizorkin [13].

Assume that $g \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\operatorname{supp} g \subset [-1, 1]$, and $\beta \in \mathbb{R}$. We consider the auxiliary function $F(x) := g(-x)e^{-i\beta x}$, $x \in \mathbb{R}$. Obviously, $F \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\operatorname{supp} F \subset [-1, 1]$. Using the function F, we construct the 2-periodic function $\psi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ (see Corollary 1). For $x \in [-2, 2]$, we have

$$\psi(x-1) = F(x-1) + F(x+1) = g(1-x)e^{-i\beta(x-1)} + g(-1-x)e^{-i\beta(x+1)}.$$

holds for all $s = 0, \dots, 2n-1$ such that $\mu_s(n,\psi) > 0$. This implies that if a function $f \in C(\mathbb{T})$ is extremal in inequality (5.8) with $p = 1$, then any function of the form $cf(t)g(t)$, where $c \in \mathbb{C}$, $g \in C(\mathbb{T})$, and $g(t) \geq 0$ for all $t \in \mathbb{R}$, is also extremal. In particular, functions of the form $h(t)g(t)$ are extremal if the function $h \in C(\mathbb{T})$ has the form (5.4), $g \in C(\mathbb{T})$, and $g(t) \geq 0$ for all $t \in \mathbb{R}$. In some sense, the converse statement holds: if the inequalities $\mu_s(n,\psi) > 0$ and $\mu_{s+1}(n,\psi) > 0$ hold for some $s \in \mathbb{Z}$, a function $f \in C(\mathbb{T})$ is extremal in inequality (5.8) with $p = 1$, and $f(t) \neq 0$ for almost all $t \in \mathbb{R}$ (with respect to the Lebesgue measure), then the function $h(t) := f(t)/|f(t)|$ belongs to $L_\infty(\mathbb{T})$ and has the form (5.4) (see the proof of Theorem 5).

We note the following well-known fact. If a function $f \in C(\mathbb{T})$ is extremal in inequality (5.8) with $p = 1$, then condition (5.10) implies that the function

$$g(u) := \int\limits_{\mathbb{T}} f(t+u)e^{-i\delta(t)}\,dt \in C(\mathbb{T})$$

is extremal in inequality (5.8) with $p = \infty$. Indeed, for all $s = 0, \dots, 2n-1$ such that $\mu_s(n,\psi) > 0$, we have $\|f\|_1 = (-1)^s g(\pi s/n) \leq \|g\|_\infty \leq \|f\|_1$ and, hence, $(-1)^s g(\pi s/n) = \|g\|_\infty$.

If $\mu_s(n,\psi) > 0$ for $s = 0, \dots, 2n-1$, then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal among trigonometric polynomials of degree at most n for which inequality (5.8) with $p = 1$ turns into an equality. Indeed, if f is an extremal trigonometric polynomial of degree at most n, then condition (5.10) is satisfied for $s = 0, \dots, 2n-1$. Then one can use the Riesz interpolation formula [16, 17] (see also [28, Ch. X, Sect. 3, (3.11)])

$$f'\left(t + \frac{\pi}{2n}\right) \equiv \sum_{s=1}^{2n} (-1)^{s-1} f\left(t + \frac{\pi s}{n}\right) a_s, \quad \text{where all } a_s > 0 \text{ and } \sum_{s=1}^{2n} a_s = n,$$

which implies the equality $\|f'\|_1 = n\|f\|_1$. Therefore, $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$ (see [3, Corollary 6]).

Remark 5. If, in Theorem 5, the function J is convex and strictly increasing on $[0, +\infty)$ and $\mu_s(n,\psi) > 0$ for all $s = 0, \dots, 2n-1$ (this implies that $\psi(0) > 0$), then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal among trigonometric polynomials of degree at most n for which inequality (5.3) turns into an equality. Indeed, if inequality (5.3) turns into an equality at some function $f \in C(\mathbb{T})$, then the corresponding inequalities (5.1) and (3.1) turn into equalities for any $t \in \mathbb{R}$ and, hence, inequality (5.8) with $p = 1$ turns into an equality at f. Then we need to use the last statement in Remark 4.

In conclusion of this section, we note that the integral inequalities (5.2) for the class of trigonometric polynomials and for different differential operators and Szegő compositions were studied by many authors, in particular, by A. Zygmund, V.V. Arestov, V.I. Ivanov, E.A. Storozhenko, V.G. Krotov, P. Oswald, and A.I. Kozko. In this case, not only convex functions J were considered. A history of this question was described in great detail in the paper by Arestov [4].

6. Generalization of Bernstein–Szegő inequalities

We denote by $\mathscr{F}_n$, $n \in \mathbb{N}$, the set of trigonometric polynomials

$$f(t) := \sum_{|k| \leq n} c_k e^{ikt} = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt), \quad c_k = c_k(f) \in \mathbb{C},$$

If a function f belongs to $C(\mathbb{T})$ and its Fourier series has the form (5.4), then, obviously, $(-1)^s f(t+\pi s/n) \equiv f(t)$ for all $s \in \mathbb{Z}$. Therefore, for such functions, we have $A_{1/n,1}(f)(t) \equiv e^{-i\beta}\psi(0) f(t+\beta/n)$ and inequality (5.3) turns into an equality.

If the function J is strictly convex at any point of the interval $(0,+\infty)$ and $\psi(0) > 0$, then Theorem 4 implies that inequality (5.3) turns into an equality at some function $f \in C(\mathbb{T}) \iff$ the functions $(-1)^s f(t+(\pi s+\beta)/n)$ are identical on $\mathbb{R}$ for all $s \in \mathbb{Z}$ such that $\mu(\{t_s\}) = c_s(\psi) > 0$ $\iff$ the functions $(-1)^s f(t+\pi s/n)$ are identical on $\mathbb{R}$ for all $s = 0, \ldots, 2n-1$ such that $\mu_s(n,\psi) > 0$. The latter equivalence is a consequence of the following properties: (1) the functions of this family with numbers $s \in \mathbb{Z}$ and $s+2nm$, $m \in \mathbb{Z}$, are identical; (2) $c_k(\psi) \geq 0$, $\mu_k(n,\psi) \geq 0$, $k \in \mathbb{Z}$, and $\mu_k(n,\psi) > 0 \iff c_{k+2nm}(\psi) > 0$ for some $m \in \mathbb{Z}$.

Assume that inequality (5.3) turns into an equality at some function $f \in C(\mathbb{T})$. If, in addition, the inequalities $\mu_s(n,\psi) > 0$ and $\mu_{s+1}(n,\psi) > 0$ hold for some $s \in \mathbb{Z}$, then, by what has been proved,

$$(-1)^s f\left(t+\frac{\pi s}{n}\right) \equiv (-1)^{s+1} f\left(t+\frac{\pi(s+1)}{n}\right).$$

Then, for the Fourier coefficients of the function f, we have the equalities $c_k(f) = -e^{ik\pi/n} c_k(f)$, $k \in \mathbb{Z}$. If $c_k(f) \neq 0$ for some $k \in \mathbb{Z}$, then $k = n(2m+1)$ for some $m \in \mathbb{Z}$. This means that the Fourier series of the function f has the form (5.4). The theorem is proved. □

Remark 4. Let $\varphi(x) \equiv e^{i\beta x}\psi(x)$, where $\beta \in \mathbb{R}$, and assume that a 2-periodic function ψ belongs to $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ and satisfies the inequality $\psi(0) > 0$. Then the operator $A_{1/n,1}$, $n \in \mathbb{N}$, satisfies the inequality (see (5.3) for $J(t) = t^p$, $1 \leq p < \infty$, or (3.5) for $\varepsilon = 1/n$, $\tau = 1$)

$$\|A_{1/n,1}(f)\|_p \leqslant \psi(0)\|f\|_p, \quad 1 \leqslant p \leqslant \infty, \quad f \in C(\mathbb{T}). \tag{5.8}$$

This inequality turns into an equality, for instance, at every function $f \in C(\mathbb{T})$ whose Fourier series has the form (5.4), since, for such functions, $A_{1/n,1}(f)(t) \equiv e^{-i\beta}\psi(0) f(t+\beta/n)$. When $1 < p < \infty$, only functions of the form (5.4) are extremal in inequality (5.8) if the inequalities $\mu_s(n,\psi) > 0$ and $\mu_{s+1}(n,\psi) > 0$ hold for some $s \in \mathbb{Z}$ (see Theorem 5 for $J(t) = t^p$). We state criteria for a function to be extremal when $p = \infty$ and $p = 1$. Taking into account Remark 2 and the fact that the Bochner measure μ of the function φ is concentrated at the points $t_k = \pi k + \beta$, $k \in \mathbb{Z}$, and $\mu(\{t_k\}) = c_k(\psi) \geq 0$, $k \in \mathbb{Z}$ (see the proof of Theorem 5), we obtain:

1) When $p = \infty$, inequality (5.8) turns into an equality at some function $f \in C(\mathbb{T})$ if and only if, for some $\eta, \delta \in \mathbb{R}$, the equality

$$(-1)^s f(\eta + \pi s/n) = e^{i\delta}\|f\|_\infty \tag{5.9}$$

holds for all $s = 0, \ldots, 2n-1$ such that $\mu_s(n,\psi) > 0$. This condition is satisfied not only for functions of the form (5.4). For instance, for $s = 0, \ldots, 2n$, we set $f(\pi s/n) := (-1)^s M$ and, at the remaining points $t \in [0, 2\pi]$, we define f so that it is continuous on $[0, 2\pi]$ with the only condition $|f(t)| \leq |M|$. For such a function f, inequality (5.8) with $p = \infty$ turns into an equality.

If $\mu_s(n,\psi) > 0$ for $s = 0, \ldots, 2n-1$, then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal among trigonometric polynomials of degree at most n for which inequality (5.8) with $p = \infty$ turns into an equality. Indeed, if f is an extremal polynomial of degree at most n, then condition (5.9) is satisfied for $s = 0, \ldots, 2n-1$ and, hence, for all $s \in \mathbb{Z}$. Then one can use the arguments of [1, Sect. 84, p. 189] for entire functions in the class B_σ with $\sigma = n$.

2) When $p = 1$, inequality (5.8) turns into an equality at some function $f \in C(\mathbb{T})$ if and only if, for any $t \in \mathbb{R}$, there exists a number $\delta(t) \in \mathbb{R}$ such that the identity

$$(-1)^s f\left(t+\frac{\pi s}{n}\right) \equiv e^{i\delta(t)}\left|f\left(t+\frac{\pi s}{n}\right)\right| \tag{5.10}$$

This implies that $A_{\varepsilon,\tau}(f)(t) = \varphi(0)c(t),\ t \in \mathbb{R}$. □

For $\varepsilon = 1/n$, $n \in \mathbb{N}$, and $\tau = 1$, we can distinguish the case where the condition on the extremal function in Theorem 4 is more clear.

Theorem 5. *Let $\varphi(x) \equiv e^{i\beta x}\psi(x)$, where $\beta \in \mathbb{R}$, and let ψ be a 2-periodic function in $\Phi(\mathbb{R}) \cap C(\mathbb{R})$. Let J be a convex nondecreasing function on $[0, +\infty)$. Then the operator $A_{1/n,1}$, $n \in \mathbb{N}$, generated by the function φ by formula* (1.1) *for $\varepsilon = 1/n$ and $\tau = 1$ satisfies the inequality*

$$\int_{\mathbb{T}} J\left(|A_{1/n,1}(f)(t)|\right)\, dt \le \int_{\mathbb{T}} J\left(\psi(0)|f(t)|\right)\, dt, \quad f \in C(\mathbb{T}). \tag{5.3}$$

Inequality (5.3) *turns into an equality, in particular, at every function $f \in C(\mathbb{T})$ whose Fourier series has the form*

$$f(t) \sim \sum_{m\in\mathbb{Z}} d_m e^{in(2m+1)t}. \tag{5.4}$$

If the function J is strictly convex at any point of the interval $(0, +\infty)$ and $\psi(0) > 0$, then inequality (5.3) *turns into an equality at some function $f \in C(\mathbb{T})$ if and only if the functions $(-1)^s f\left(t + \frac{\pi s}{n}\right)$ are identical on $\mathbb{R}$ for all $s = 0, \dots, 2n-1$ such that $\mu_s(n, \psi) > 0$, where*

$$\mu_k(n, \psi) = \sum_{m\in\mathbb{Z}} c_{k+2nm}(\psi), \quad k \in \mathbb{Z}, \tag{5.5}$$

and $c_k(\psi) \ge 0$, $k \in \mathbb{Z}$, are the Fourier coefficients of the function ψ. If, in addition, the inequalities $\mu_s(n, \psi) > 0$ and $\mu_{s+1}(n, \psi) > 0$ hold for some $s \in \mathbb{Z}$, then inequality (5.3) *turns into an equality only at functions $f \in C(\mathbb{T})$ whose Fourier series has the form* (5.4).

P r o o f. In our case, $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\varphi(0) = \psi(0)$. Therefore, inequality (5.3) follows immediately from inequality (5.2).

Since the function ψ belongs to $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ and is 2-periodic, its Fourier coefficients $c_k(\psi)$, $k \in \mathbb{Z}$, are nonnegative and ψ is expanded into an absolutely convergent Fourier series. Then the function φ is also expanded into an absolutely convergent series:

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k(\psi) e^{i(\pi k+\beta)x}, \quad x \in \mathbb{R}.$$

It follows from this representation that the Bochner measure μ of the function φ is concentrated at the points $t_k = \pi k + \beta$, $k \in \mathbb{Z}$, and $\mu(\{t_k\}) = c_k(\psi)$, $k \in \mathbb{Z}$. Therefore, for any $f \in C(\mathbb{T})$, we have

$$A_{1/n,1}(f)(t) = e^{-i\beta} \sum_{k\in\mathbb{Z}} (-1)^k f\left(t + \frac{t_k}{n}\right) c_k(\psi), \quad t \in \mathbb{R}.$$

Taking into account the periodicity of f, it is convenient to divide the terms in this sum into disjoint groups in which the summation index has the form $k + 2nm$ with $m \in \mathbb{Z}$ and $k = 0, \dots, 2n-1$. Then

$$A_{1/n,1}(f)(t) = e^{-i\beta} \sum_{k=0}^{2n-1} (-1)^k f\left(t + \frac{\pi k + \beta}{n}\right) \mu_k(n, \psi), \quad t \in \mathbb{R}, \tag{5.6}$$

where the numbers $\mu_k(n, \psi)$ are defined by formula (5.5). For these numbers, the following equalities hold:

$$\sum_{k=0}^{2n-1} \mu_k(n, \psi) = \sum_{k\in\mathbb{Z}} c_k(\psi) = \psi(0); \quad \mu_k(n, \psi) = \mu_{k+2n}(n, \psi), \quad k \in \mathbb{Z}. \tag{5.7}$$

Successively using the monotonicity and the Jensen inequality (see, for instance, [12, Sect. 2.2] or Proposition 2), for $f \in C(\mathbb{T})$, we derive from inequality (3.1) that

$$\begin{aligned} J\left(\frac{1}{\varphi(0)}|A_{\varepsilon,\tau}(f)(t)|\right) &\le J\left(\frac{1}{\varphi(0)}\int\limits_{\mathbb{R}}|f(t+\varepsilon u)|d\mu(u)\right) \\ &\le \frac{1}{\varphi(0)}\int\limits_{\mathbb{R}} J\left(|f(t+\varepsilon u)|\right)d\mu(u), \quad t\in\mathbb{R}. \end{aligned} \tag{5.1}$$

We integrate the left-hand and right-hand sides of inequality (5.1) with respect to $t \in \mathbb{T}$. Applying the Fubini theorem and taking into account the periodicity of f, we obtain

$$\int\limits_{\mathbb{T}} J\left(\frac{1}{\varphi(0)}|A_{\varepsilon,\tau}(f)(t)|\right)dt \le \int\limits_{\mathbb{T}} J\left(|f(t)|\right)dt.$$

In view of the arbitrariness of f, it is convenient to write the latter inequality in the form

$$\int\limits_{\mathbb{T}} J\left(|A_{\varepsilon,\tau}(f)(t)|\right)dt \le \int\limits_{\mathbb{T}} J\left(\varphi(0)|f(t)|\right)dt. \tag{5.2}$$

Inequality (5.2) also holds if $\varphi(0) = 0$, since, in this case, $\varphi(x) \equiv 0$ and, hence, $A_{\varepsilon,\tau}(f)(t) \equiv 0$ for any $f \in C(\mathbb{T})$. Thus, we obtain the following theorem.

Theorem 4. *Assume that $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\tau, \varepsilon \in \mathbb{R}$, $\varepsilon \ne 0$, and J is a convex nondecreasing function on $[0, +\infty)$. Then:*

1) *The operator $A_{\varepsilon,\tau}$ generated by the function φ by formula* (1.1) *satisfies inequality* (5.2) *for any function $f \in C(\mathbb{T})$.*

2) *If the condition $|\varphi(\varepsilon s - \tau)| = \varphi(0)$ holds for some $s \in \mathbb{Z}$, then equality in* (5.2) *is attained at the polynomials $f(t) = ce^{ist}$, $c \in \mathbb{C}$. If $\tau/\varepsilon \in \mathbb{Z}$, then this condition holds for $s = \tau/\varepsilon$.*

If condition (3.4) *holds for some $s, m \in \mathbb{Z}$, $s \ne m$, then equality in* (5.2) *is attained at the polynomials $f(t) = ce^{ist} + \nu e^{imt}$, $c, \nu \in \mathbb{C}$.*

If $\tau \ne 0$, $|\varphi(-2\tau)| = \varphi(0)$, $\varepsilon = \tau/n$, and $n \in \mathbb{N}$, then equality in (5.2) *is attained at the polynomials $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$.*

3) *If the function J is strictly convex at any point of the interval $(0, +\infty)$ and $\varphi(0) > 0$, then inequality* (5.2) *turns into an equality at some function $f \in C(\mathbb{T})$ if and only if, for any $t \in \mathbb{R}$ and μ-almost all $u \in \mathbb{R}$, the equality $e^{-iu\tau}f(t + \varepsilon u) = c(t)$ holds, where $c(t) = A_{\varepsilon,\tau}(f)(t)/\varphi(0) \in C(\mathbb{T})$.*

P r o o f. Only the latter statement needs to be proved. The sufficiency is obvious. Let us prove the necessity. Let inequality (5.2) turn into an equality for some function $f \in C(\mathbb{T})$. Then inequalities (5.1) turn into equalities for all $t \in \mathbb{R}$. Let

$$\alpha(t) := \frac{1}{\varphi(0)}\int\limits_{\mathbb{R}}|f(t+\varepsilon u)|d\mu(u), \quad t\in\mathbb{R}.$$

Obviously, $\alpha(t) \ge 0$ for all $t \in \mathbb{R}$. If $\alpha(t) = 0$, then $f(t+\varepsilon u) = 0$ for μ-almost all $u \in \mathbb{R}$ and, in this case, $c(t) = 0$. If $\alpha(t) > 0$, then $|f(t+\varepsilon u)| = \alpha(t)$ for μ-almost all $u \in \mathbb{R}$ (see Proposition 2). Since the function J strictly increases on $[0, +\infty)$, inequality (3.1) also turns into an equality for all $t \in \mathbb{R}$. Therefore, for some $\beta(t) \in \mathbb{R}$ and μ-almost all $u \in \mathbb{R}$, we have the equality (see Proposition 1)

$$e^{-iu\tau}f(t+\varepsilon u) = e^{i\beta(t)}|e^{-iu\tau}f(t+\varepsilon u)| = e^{i\beta(t)}\alpha(t) = c(t).$$

$c(t) = A_{\varepsilon,\tau}(f)(t)/\varphi(0) \in C(\mathbb{T})$ (for such p, see Theorem 4 below for $J(t) = t^p$).

Remark 3. If $1 \le p < \infty$, the class $C(\mathbb{T})$ is everywhere dense in $L_p(\mathbb{T})$ (the Lebesgue measure is taken as a measure). Therefore, inequality (3.5) implies that the multiplier $A_{\varepsilon,\tau} : C(\mathbb{T}) \to C(\mathbb{T})$ defined by formula (3.2) is extended to the multiplier $A_{\varepsilon,\tau} : L_p(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p < \infty$, and

$$\|A_{\varepsilon,\tau}(f)\|_p \leqslant \varphi(0)\|f\|_p, \quad 1 \leqslant p < \infty, \quad f \in L_p(\mathbb{T}). \tag{3.6}$$

Hence, $A_{\varepsilon,\tau} : L_\infty(\mathbb{T}) \to L_\infty(\mathbb{T})$ and inequality (3.6) holds with $p = \infty$. We only need to use the well-known facts from measure and integration theory (see Proposition 3).

4. Periodic positive definite functions

The following description of periodic functions of the class $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ is well known (see, for instance, [7, Theorem 1.7.5] and [10, Sect. II.1]).

Theorem 3. *If $\psi \in C(\mathbb{R})$ and ψ is $2T$-periodic with $T > 0$, then $\psi \in \Phi(\mathbb{R})$ if and only if $c_k(\psi) \ge 0$, $k \in \mathbb{Z}$, where*

$$c_k(\psi) := \frac{1}{2T} \int_{-T}^{T} \psi(x) e^{-i\pi kx/T} dx, \quad k \in \mathbb{Z}.$$

In this case, the function ψ is expanded into the absolutely convergent Fourier series

$$\psi(x) = \sum_{k\in\mathbb{Z}} c_k(\psi) e^{i\pi kx/T}, \quad x \in \mathbb{R}.$$

Corollary 1. *Assume that $f \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\operatorname{supp} f \subset [-1, 1]$, and a 2-periodic function $\psi(x)$ coincides with the function $f(x)$ for $x \in [-1, 1]$. Then $\psi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\psi(x - 1) = f(x - 1) + f(x + 1)$ for $x \in [-2, 2]$.*

P r o o f. Since $\psi(\pm 1) = f(\pm 1) = 0$, we have $\psi \in C(\mathbb{R})$ and

$$2c_k(\psi) = \int_{-1}^{1} f(x) e^{-i\pi kx} dx = \widehat{f}(\pi k) \ge 0, \quad k \in \mathbb{Z}.$$

Therefore, $\psi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$. Since $\operatorname{supp} f \subset [-1, 1]$, we obviously have

$$\psi(x - 1) = \sum_{k\in\mathbb{Z}} f(x - 1 + 2k), \quad x \in \mathbb{R}.$$

Only terms with $k = 0$ and $k = 1$ remain in this sum for $x \in [-2, 2]$. □

5. Sharp integral inequalities for periodic functions

Let $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\varphi(0) > 0$. Assume that J is a convex nondecreasing function on $[0, +\infty)$. Then J is continuous on $[0, +\infty)$ and can be extended to $\mathbb{R}$ with preservation of convexity (for instance, by defining $J(t) := J(0)$ for $t < 0$ or by means of the even extension).

If, for some $s, m \in \mathbb{Z}$, $s \neq m$, we have

$$|\varphi(\varepsilon s - \tau)| = |\varphi(\varepsilon m - \tau)| = \varphi(0), \tag{3.4}$$

then equality (3.3) holds for the polynomial $f(t) = ce^{ist} + \nu e^{imt}$, $c, \nu \in \mathbb{C}$, since, in this case,

$$A_{\varepsilon,\tau}(f)(t) = \varphi(\varepsilon s - \tau)ce^{ist} + \varphi(\varepsilon m - \tau)\nu e^{imt}.$$

We only need to take into account that, for any $\delta, \alpha \in \mathbb{R}$, the following equalities hold:

$$\left\|ce^{ist} + e^{i\delta}\nu e^{imt}\right\|_p = \left\|ce^{is(t+\alpha)} + e^{i\delta}\nu e^{im(t+\alpha)}\right\|_p = \left\|ce^{ist} + e^{i(\delta+m\alpha-s\alpha)}\nu e^{imt}\right\|_p.$$

In particular, the latter equality holds for $\alpha = \delta/(s-m)$.

If $\tau \neq 0$, $|\varphi(-2\tau)| = \varphi(0)$, $\varepsilon = \tau/n$, and $n \in \mathbb{N}$, then condition (3.4) is satisfied for $s = n$ and $m = -n$. Hence, $\|A_{\tau/n,\tau}(f)\|_p = \varphi(0)\|f\|_p$ for the polynomial $f(t) = ce^{int} + \nu e^{-int}$ with $c, \nu \in \mathbb{C}$.

Thus, we have proved the following theorem.

Theorem 2. *Assume that $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\tau, \varepsilon \in \mathbb{R}$, and $\varepsilon \neq 0$. Then:*

1) *the operator $A_{\varepsilon,\tau}$ acts from $C(\mathbb{T})$ to $C(\mathbb{T})$, is a multiplier, and satisfies the inequality*

$$\|A_{\varepsilon,\tau}(f)\|_p \leqslant \varphi(0)\|f\|_p, \quad 1 \leqslant p \leqslant \infty, \quad f \in C(\mathbb{T}); \tag{3.5}$$

2) *if, for some $s \in \mathbb{Z}$, the condition $|\varphi(\varepsilon s - \tau)| = \varphi(0)$ is satisfied, then equality in (3.5) is attained at the polynomials $f(t) = ce^{ist}$, $c \in \mathbb{C}$. If $\tau/\varepsilon \in \mathbb{Z}$, this condition is satisfied for $s = \tau/\varepsilon$.*

If, for some $s, m \in \mathbb{Z}$, $s \neq m$, condition (3.4) is satisfied, then equality in (3.5) is attained at the polynomials $f(t) = ce^{ist} + \nu e^{imt}$, $c, \nu \in \mathbb{C}$.

If $\tau \neq 0$ and $|\varphi(-2\tau)| = \varphi(0)$, then equality in (3.5) for $\varepsilon = \tau/n$, $n \in \mathbb{N}$, is attained at the polynomials $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$.

Remark 1. In connection with the conditions in Theorem 2, the following simple property of positive definite functions is useful: if $\varphi \in \Phi(\mathbb{R})$ and, for some $y, \delta \in \mathbb{R}$, $y \neq 0$, we have $\varphi(y) = \varphi(0)e^{i\delta y}$, then $\varphi(x) \equiv f(x)e^{i\delta x}$, where $f \in \Phi(\mathbb{R})$ and f is periodic with period $|y| > 0$. Indeed, the function $f(x) \equiv \varphi(x)e^{-i\delta x}$ is the product of two positive definite functions. Therefore, $f \in \Phi(\mathbb{R})$ and, hence, for any $x \in \mathbb{R}$, we have

$$|f(x+y) - f(x)|^2 \leq 2f(0)(f(0) - \mathrm{Re} f(y)).$$

Since $f(y) = \varphi(y)e^{-i\delta y} = \varphi(0) = f(0) \geq 0$, we have $f(x+y) - f(x) = 0$ for all $x \in \mathbb{R}$. If, in addition, $\varphi \in C(\mathbb{R})$, then the Bochner measure of the function φ is discrete and concentrated at the points $t_k = 2\pi k/|y| + \delta$, $k \in \mathbb{Z}$, and $\mu(\{t_k\}) = c_k(f) \geq 0$, $k \in \mathbb{Z}$ (see Theorem 3 below).

Remark 2. When $p = \infty$, inequality (3.5) turns into an equality at some function $f \in C(\mathbb{T})$ (see inequality (3.1) and Proposition 1) if and only if the equality $f(\xi + \varepsilon u) = e^{i(u\tau+\beta)}\|f\|_\infty$ holds for some $\xi, \beta \in \mathbb{R}$ and μ-almost all $u \in \mathbb{R}$.

When $p = 1$, inequality (3.5) turns into an equality at some function $f \in C(\mathbb{T})$ (see inequality (3.1) and Proposition 1) if and only if, for any $t \in \mathbb{R}$, there exists a number $\beta(t) \in \mathbb{R}$ such that the equality $f(t + \varepsilon u) = e^{i(u\tau+\beta(t))}|f(t+\varepsilon u)|$ holds for μ-almost all $u \in \mathbb{R}$. This implies that if a function $f \in C(\mathbb{T})$ is extremal in inequality (3.5) with $p = 1$, then any function of the form $cf(t)g(t)$, where $c \in \mathbb{C}$, $g \in C(\mathbb{T})$, and $g(t) \geq 0$ for all $t \in \mathbb{R}$, is also extremal.

When $p \in (1, \infty)$, inequality (3.5) turns into an equality at some function $f \in C(\mathbb{T})$ if and only if, for any $t \in \mathbb{R}$ and μ-almost all $u \in \mathbb{R}$, the equality $f(t + \varepsilon u) = e^{iu\tau}c(t)$ holds, where

P r o o f. (i) We take a sequence $\{p_n\}$, $n \in \mathbb{N}$, such that $p_n > 0$, $p_n \to +\infty$, and $\|f\|_{p_n} \to c := \liminf_{p\to+\infty} \|f\|_p \geq 0$. For an arbitrary $\sigma > c$, we define $\varepsilon := (\sigma - c)/2 > 0$. Then there exists a number $n(\sigma)$ such that the inequality $\|f\|_{p_n} \leq c + \varepsilon = (\sigma + c)/2 < \sigma$ holds for all $n \geqslant n(\sigma)$. The Chebyshev inequality implies that

$$\mu(\{x \in \Omega : |f(x)| \geq \sigma\}) \leq \left(\frac{\|f\|_{p_n}}{\sigma}\right)^{p_n} \to 0, \quad n \to +\infty.$$

Therefore, $|f(x)| < \sigma$ for μ-almost all $x \in \Omega$ and, hence, $\|f\|_\infty \leq c$.

(ii) If $\|f\|_q = 0$, the required assertion is obvious. Let $\|f\|_q > 0$. Then, for any $p > q$, the inequality $\|f\|_p \leq \|f\|_\infty^{(p-q)/p} \|f\|_q^{q/p}$ holds. This inequality and assertion (i) yield

$$\limsup_{p\to+\infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p\to+\infty} \|f\|_p \leq \limsup_{p\to+\infty} \|f\|_p\,.$$

□

3. Sharp L_p-inequalities for periodic functions

Equality (1.1) implies the inequality

$$|A_{\varepsilon,\tau}(f)(t)| \leq \int_{\mathbb{R}} |f(t + \varepsilon u)| d\mu(u), \quad f \in C(\mathbb{T}), \quad t \in \mathbb{R}. \tag{3.1}$$

Obviously, $\|A_{\varepsilon,\tau}(f)\|_\infty \leqslant \varphi(0)\|f\|_\infty$.

If $1 \leqslant p < \infty$, then inequality (3.1) along with the Minkowski inequality [12, Theorem 2.4] yields

$$\begin{aligned}\|A_{\varepsilon,\tau}(f)\|_p &= \Big(\int_{\mathbb{T}} |A_{\varepsilon,\tau}(f)|^p \, dt\Big)^{\frac{1}{p}} \leqslant \bigg(\int_{\mathbb{T}} \Big(\int_{\mathbb{R}} |f(t + \varepsilon u)| d\mu(u)\Big)^p dt\bigg)^{\frac{1}{p}} \\ &\leqslant \int_{\mathbb{R}} \Big(\int_{\mathbb{T}} |f(t + \varepsilon u)|^p dt\Big)^{\frac{1}{p}} d\mu(u) = \varphi(0)\|f\|_p.\end{aligned}$$

It follows from the Fubini theorem that the Fourier series of the function $A_{\varepsilon,\tau}(f)(t)$ has the form

$$A_{\varepsilon,\tau}(f)(t) \sim \sum_{k\in\mathbb{Z}} \varphi(\varepsilon k - \tau) c_k(f) e^{ikt}, \quad f \in C(\mathbb{T}), \tag{3.2}$$

where $c_k(f)$ are the Fourier coefficients of the function f:

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt, \quad k \in \mathbb{Z}\,.$$

Let us find sufficient conditions for the equality

$$\|A_{\varepsilon,\tau}(f)\|_p = \varphi(0)\|f\|_p. \tag{3.3}$$

If $|\varphi(\varepsilon s - \tau)| = \varphi(0)$ for some $s \in \mathbb{Z}$, then equality (3.3) holds for the polynomial $f(t) = ce^{ist}$, $c \in \mathbb{C}$, since, in this case, $A_{\varepsilon,\tau}(f)(t) = \varphi(\varepsilon s - \tau)ce^{ist}$. If $\tau/\varepsilon \in \mathbb{Z}$, this condition is satisfied for $s = \tau/\varepsilon$.

2. Auxiliary facts of measure and integration theory

We recall some well-known facts which are used in the paper to describe extremal functions. In this section, a measure μ is a nonnegative countably additive function defined on a σ-algebra γ with identity element Ω. For $p \in (0, +\infty)$, the class $L_p(\Omega, \gamma, \mu)$ is the set of all γ-measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_p := \Bigg(\int\limits_\Omega |f(u)|^p \, d\mu(u) \Bigg)^{1/p} < +\infty.$$

The class $L_\infty(\Omega, \gamma, \mu)$ is the set of all γ-measurable functions $f : \Omega \to \mathbb{C}$ for which there exists $K = K(f) < +\infty$ such that $|f(u)| \le K$ for μ-almost every $u \in \Omega$. For $f \in L_\infty(\Omega, \gamma, \mu)$, the norm is defined by the formula

$$\|f\|_\infty := \inf\{K : |f(u)| \le K \text{ for } \mu\text{-almost all } u \in \Omega\}.$$

For convenience, we assume that $L_p(\Omega, \gamma, \mu) = L_p(\Omega, \mu) = L_p(\Omega)$.

Proposition 1. *Let (Ω, γ, μ) be a measurable space with measure. If $f \in L_1(\Omega, \mu)$, then*

$$\Bigg| \int\limits_\Omega f(u) \, d\mu(u) \Bigg| \le \int\limits_\Omega |f(u)| \, d\mu(u)$$

and the inequality turns into an equality if and only if the equality $f(u) = e^{i\theta}|f(u)|$ holds for some $\theta \in \mathbb{R}$ and for μ-almost all $u \in \Omega$.

P r o o f. See, for instance, [18, Theorems 1.33 and 1.39]. Obviously, for some $\beta \in \mathbb{R}$, we have

$$\Bigg| \int\limits_\Omega f(u) \, d\mu(u) \Bigg| = e^{i\beta} \int\limits_\Omega f(u) \, d\mu(u) = \int\limits_\Omega e^{i\beta} f(u) \, d\mu(u) = \int\limits_\Omega \mathrm{Re}(e^{i\beta} f(u)) \, d\mu(u) \le \int\limits_\Omega |f(u)| \, d\mu(u)$$

and the inequality turns into an equality if and only if $\mathrm{Re}(e^{i\beta} f(u)) = |f(u)|$ for μ-almost all $u \in \Omega$ or if and only if $e^{i\beta} f(u) = |f(u)|$ for μ-almost all $u \in \Omega$. $\square$

Proposition 2. *Assume that J is a convex function on $\mathbb{R}$, (Ω, γ, μ) is a measurable space with finite measure, $\mu(\Omega) > 0$, and f is a real-valued function in $L_1(\Omega, \mu)$. Then*

$$J\Bigg(\frac{1}{\mu(\Omega)} \int\limits_\Omega f(u) \, d\mu(u) \Bigg) \le \frac{1}{\mu(\Omega)} \int\limits_\Omega J(f(u)) \, d\mu(u). \tag{2.1}$$

If the function J is strictly convex at the point $\alpha = \int_\Omega f(u) \, d\mu(u)/\mu(\Omega)$, then equality in (2.1) is attained if and only if $f(u) = \alpha$ for μ-almost all $u \in \Omega$.

For a proof of this result, see, for instance, [12, Sect. 2.2].

The next proposition will be needed only in Remark 3.

Proposition 3. *Let (Ω, γ, μ) be a measurable space with measure. Then*:

(i) *if, for some $q > 0$, we have $f \in L_p(\Omega)$ for all $p \in [q, +\infty)$ and $\liminf\limits_{p \to +\infty} \|f\|_p < +\infty$, then $f \in L_\infty(\Omega)$ and $\|f\|_\infty \le \liminf\limits_{p \to +\infty} \|f\|_p$;*

(ii) *if, for some $q > 0$, we have $f \in L_\infty(\Omega) \cap L_q(\Omega)$, then $f \in L_p(\Omega)$ for all $p \in [q, +\infty)$ and $\|f\|_\infty = \lim\limits_{p \to +\infty} \|f\|_p$.*

$\overline{f(-x)} = f(x)$, $|f(x+y)-f(x)|^2 \le 2f(0)(f(0)-\mathrm{Re}f(y))$, $x, y \in \mathbb{R}$, and $\bar{f}$, $\mathrm{Re}\, f$, $fg \in \Phi(\mathbb{R})$. In 1932, S. Bochner and, independently, A. Khinchin proved the following criterion of positive definiteness.

Theorem 1 (Bochner–Khinchin). *The inclusion $f \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ holds if and only if there exists a finite nonnegative Borel measure μ on $\mathbb{R}$ such that*

$$f(x) = \int_{\mathbb{R}} e^{ixt} d\mu(t), \quad x \in \mathbb{R}.$$

The proof of this theorem can be found, for instance, in [2, 7, 19, 23, 24]. As a direct consequence, we obtain the following criterion of positive definiteness in terms of nonnegativity of the Fourier transform: *if $f \in C(\mathbb{R}) \cap L_1(\mathbb{R})$, then $f \in \Phi(\mathbb{R}) \iff \widehat{f}(t) \ge 0$, $t \in \mathbb{R}$, where*

$$\widehat{f}(t) := \int_{\mathbb{R}} e^{-itx} f(x) dx, \quad t \in \mathbb{R}.$$

Using this criterion, it is not difficult to see that the functions $(1-|x|)_+$, $e^{-|x|}$, and e^{-x^2} are positive definite.

We denote by $C(\mathbb{T})$, $\mathbb{T} := [-\pi, \pi]$, the class of 2π-periodic continuous functions $f : \mathbb{R} \to \mathbb{C}$. For $f \in C(\mathbb{T})$, we define

$$\|f\|_\infty := \sup\{|f(t)| : \ t \in \mathbb{T}\} \ \text{and} \ \|f\|_p := \left(\int_{\mathbb{T}} |f(t)|^p dt \right)^{1/p}, \quad 1 \le p < \infty.$$

Let $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, and let μ be the corresponding finite nonnegative Borel measure on $\mathbb{R}$ such that

$$\varphi(x) = \int_{\mathbb{R}} e^{ixu} d\mu(u), \quad x \in \mathbb{R}.$$

For fixed $\varepsilon, \tau \in \mathbb{R}$, $\varepsilon \ne 0$, we consider the linear operator $A_{\varepsilon,\tau}$ generated by the function φ:

$$A_{\varepsilon,\tau}(f)(t) := \int_{\mathbb{R}} e^{-iu\tau} f(t + \varepsilon u) d\mu(u), \quad t \in \mathbb{R}, \quad f \in C(\mathbb{T}). \tag{1.1}$$

The function $A_{\varepsilon,\tau}(f)(t)$ is continuous on $\mathbb{R}$ and 2π-periodic. Therefore, $A_{\varepsilon,\tau} : C(\mathbb{T}) \to C(\mathbb{T})$. In this paper, we prove the inequalities

$$\|A_{\varepsilon,\tau}(f)\|_p \leqslant \varphi(0)\|f\|_p, \quad \int_{\mathbb{T}} J\left(|A_{\varepsilon,\tau}(f)(t)|\right) dt \le \int_{\mathbb{T}} J\left(\varphi(0)|f(t)|\right) dt,$$

where $1 \leqslant p \leqslant \infty$, $f \in C(\mathbb{T})$, and J is a convex nondecreasing function on $[0, +\infty)$. In addition, we obtain some criteria of extremal function in these inequalities (see Theorems 2 and 4 and Remark 2). We study in more detail the case in which $\varepsilon = 1/n$, $n \in \mathbb{N}$, $\tau = 1$, and $\varphi(x) \equiv e^{i\beta x}\psi(x)$, where $\beta \in \mathbb{R}$ and ψ is a 2-periodic function of the class $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ (see Theorem 5 and Remarks 4 and 5). In turn, we consider in more detail the case where a 2-periodic function ψ is constructed by means of a finite function $g \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ (Theorem 6). As a particular case, we obtain the Berstein–Szegő inequality for the Weyl–Nagy derivative of trigonometric polynomials (Remark 7). In Theorem 8, we consider the case of the family of functions $g_{1/n,h}(x) := hg(x) + (1 - 1/n - h)g(nx)$, where $n \in \mathbb{N}$, $n \ge 2$, $-1/n \le h \le 1 - 1/n$, and the function $g \in C(\mathbb{R})$ is even, nonnegative, decreasing, and convex on $(0, +\infty)$ with $\mathrm{supp}\, g \subset [-1, 1]$. This case is related to the positive definiteness of piecewise linear functions [15]. In Theorem 9 and Corollary 3, we obtain general interpolation formulas for periodic functions which include the known interpolation formulas of M. Riesz, of G. Szegő, and of A.I. Kozko [11] for trigonometric polynomials (see Remark 8).

POSITIVE DEFINITE FUNCTIONS AND SHARP INEQUALITIES FOR PERIODIC FUNCTIONS

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Abstract: Let φ be a positive definite and continuous function on $\mathbb{R}$, and let μ be the corresponding Bochner measure. For fixed $\varepsilon, \tau \in \mathbb{R}$, $\varepsilon \neq 0$, we consider a linear operator $A_{\varepsilon,\tau}$ generated by the function φ:

$$A_{\varepsilon,\tau}(f)(t) := \int_{\mathbb{R}} e^{-iu\tau} f(t + \varepsilon u) d\mu(u), \quad t \in \mathbb{R}, \quad f \in C(\mathbb{T}).$$

Let J be a convex and nondecreasing function on $[0, +\infty)$. In this paper, we prove the inequalities

$$\|A_{\varepsilon,\tau}(f)\|_p \leqslant \varphi(0)\|f\|_p, \quad \int_{\mathbb{T}} J\left(|A_{\varepsilon,\tau}(f)(t)|\right) dt \leq \int_{\mathbb{T}} J\left(\varphi(0)|f(t)|\right) dt$$

for $p \in [1, \infty]$ and $f \in C(\mathbb{T})$ and obtain criteria of extremal function. We study in more detail the case in which $\varepsilon = 1/n$, $n \in \mathbb{N}$, $\tau = 1$, and $\varphi(x) \equiv e^{i\beta x}\psi(x)$, where $\beta \in \mathbb{R}$ and the function ψ is 2-periodic and positive definite. In turn, we consider in more detail the case where the 2-periodic function ψ is constructed by means of a finite positive definite function g. As a particular case, we obtain the Bernstein–Szegő inequality for the derivative in the Weyl–Nagy sense of trigonometric polynomials. In one of our results, we consider the case of the family of functions $g_{1/n,h}(x) := hg(x) + (1 - 1/n - h)g(nx)$, where $n \in \mathbb{N}$, $n \geq 2$, $-1/n \leq h \leq 1 - 1/n$, and the function $g \in C(\mathbb{R})$ is even, nonnegative, decreasing, and convex on $(0, +\infty)$ with $\operatorname{supp} g \subset [-1, 1]$. This case is related to the positive definiteness of piecewise linear functions. We also obtain some general interpolation formulas for periodic functions and trigonometric polynomials which include the known interpolation formulas of M. Riesz, of G. Szegő, and of A.I. Kozko for trigonometric polynomials.

Key words: Positive definite function, Trigonometric polynomial, Weyl–Nagy derivative, Bernstein–Szegő inequality, Interpolation formula.

1. Introduction

The role of positive definite functions in obtaining sharp inequalities for trigonometric polynomials and entire functions is well known (see, for instance, Boas [6, Ch. 11], Timan [22, Sect. 4.8], Lizorkin [13], Gorin [9], and Trigub and Belinsky [23]). For instance, the classical Bernstein inequality $\max|f'(x)| \leq n \max|f(x)|$ for trigonometric polynomials of degree at most n is related to the positive definiteness of the function $(1 - |x|)_+$. A historical survey of such inequalities and the methods of their proof are given in the works by Gorin [9], Arestov and Glazyrina [5], Gashkov [8], and Vinogradov [25]. In the present paper, we obtain sharp inequalities for continuous periodic functions and, in particular, for trigonometric polynomials. These inequalities are related to positive definite functions. As consequences, we obtain generalizations of Bernstein–Szegő inequalities. We give criteria and descriptions of extremal functions in these inequalities.

A complex-valued function $f : \mathbb{R} \to \mathbb{C}$ is called positive definite on $\mathbb{R}$ ($f \in \Phi(\mathbb{R})$) if, for any $m \in \mathbb{N}$, any set of points $\{x_k\}_{k=1}^m \subset \mathbb{R}$, and any complex numbers $\{c_k\}_{k=1}^m \subset \mathbb{C}$, the following inequality holds:

$$\sum_{k,j=1}^{m} c_k \overline{c_j} f(x_k - x_j) \geqslant 0.$$

It is easy to verify that, for any $\beta \in \mathbb{R}$, the function $f(x) = e^{i\beta x}$ is positive definite. For a function in $\Phi(\mathbb{R})$, the continuity at zero is equivalent to the continuity on $\mathbb{R}$. If $f, g \in \Phi(\mathbb{R})$, then $|f(x)| \leq f(0)$,

REFERENCES

1. **Smirnov V.I., Lebedev N.A.** Constructive theory of functions of complex variables. Moscow-Leningrad: Nauka, 1964 [in Russian].
2. **Abilov V.A., Abilova F.V., Kerimov M.K.** Sharp estimates for the rate of convergence of Fourier series of functions of a complex variable in the space $L_2(D, p(z))$ // J. Comp. Math. and Math. Physics, 2010. Vol. 50, no 6. P. 999–1004. [in Russian]
3. **Tukhliev K.** Mean-square approximation of functions by Fourier–Bessel series and the values of widths for some functional classes // Chebyshevskii Sb., 2016. Vol. 17, no. 4. P. 141–156. [in Russian]
4. **Vakarchuk S.B.** Mean approximation of functions on the real axis by algebraic polynomials with Chebyshev–Hermite weight and widths of function class // Math. zametki, 2014. Vol. 95, no. 5. P. 666–684. [in Russian]
5. **Shabozov M.Sh., Tukhliev K.** Jackson–Stechkin type inequalities with generalized moduli of continuity and widths of some classes of functions // Proceedings of Instit. of Math. and Mech. Ural Branch of RAS, 2015. Vol. 21, no. 4. P. 292–308.
6. **Bitsadze A.V.** Fundamentals of the theory of analytic functions of a complex variables. Moscow: Nauka, 1984. [in Russian]
7. **Berg J., Lofstrom J.** Interpolation spaces. An introduction. Moscow: Mir, 1980. [in Russian]
8. **Ditzian Z., Totik V.** K-functionals and best polynomial approximation in weighted $L^p(\mathbb{R})$ // J. Approx. Theory, 1986. Vol. 46, no. 1. P. 38–41.
9. **Ditzian Z., Totik V.** Moduli of smoothness. Berlin: Springer-Verlag. Heidelberg. New York. Tokyo, 1987.
10. **Pinkus A.** n-widths in approximation theory. Berlin: Springer-Verlag. Heidelberg. New York. Tokyo, 1985.
11. **Tikhomirov V.M.** Some questions of approximation theory. Moscow: MSU, 1976. [in Russian]
12. **Shevchuk I.A.** Approximation of polynomials and traces of continuous functions on a closed interval. Kiev: Naukova Dumka, 1992. [Russian]
13. **Farkov Yu.A.** Widths of Hardy classes and Bergman classes on the ball in $\mathbb{C}^n$ // Uspekhi Mat. Nauki, 1990. Vol. 45, no. 5. P. 229–231. [in Russian]
14. **Farkov Yu.A.** The n-widths of Hardy-Sobolev spaces of several complex variables // J. Approx. Theory, 1993. Vol.75, no. 2. P. 183–197.
15. **Vakarchuk S.B.** Best linear methods of approximation and widths of classes of analytic functions in a disk // Math. zametki, 1995. Vol. 57, no. 1. P. 30–39. [in Russian]
16. **Vakarchuk S.B., Shabozov M.Sh.** The widths of classes of analytic functions in a disc // Mat. Sb., 2010. Vol. 201, no. 8. P. 3–22 [in Russian].
17. **Shabozov M.Sh., Langarshoev M.R.** The best linear methods and values of widths for some classes of analytic functions in the Bergman weight space // Dokl. Akad. Nauk, 2013. Vol. 450, no. 5. P. 518–521. [in Russian]
18. **Shabozov M.Sh., Saidusaynov M.S.** The values of n-widths and the best linear method of approximation of some classes of functions in the weighted Bergman space // Izv. TSU. Natural Science, 2014. No. 3. P. 40–57.
19. **Saidusaynov M.S.** On the best linear method of approximation of some classes analytic functions in the weighted Bergman space // Chebyshevskii Sb., 2016. Vol. 17, no. 1. P. 240–253. [in Russian]

and
$$\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}\le t\le 1.$$
First, assume that
$$0<t\le\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}.$$
In this case, using inequality (22) with
$$t_1=t,\quad t_2=\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}$$
and applying (12) and (21), for any $p_n\in\mathcal{M}_{n+1}$, we obtain
$$\begin{aligned}&\mathcal{K}_m(p_n^{(r)},t^m)_2\le t^m\cdot\|p_n^{(r+m)}\|\le t^m\cdot\alpha_{n,r}\cdot\alpha_{n-r,m}\|p_n\|\\&\le t^m\cdot\alpha_{n,r}\cdot\alpha_{n-r,m}\cdot\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\le t^m\cdot\alpha_{n-r,m}\cdot\sqrt{\frac{n-r+1}{n-r-m+1}}\cdot\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\le\Psi(t^m).\end{aligned}\tag{23}$$

Now, let
$$\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}\le t\le 1.$$
Then using (16) and the Bernstein type inequality
$$\|p_n^{(r)}\|\le\alpha_{n,r}\cdot\|p_n\|$$
and taking into account that the majorant Ψ is nondecreasing, we find that
$$\begin{aligned}&\mathcal{K}_m(p_n^{(r)},t^m)_2\le\|p_n^{(r)}\|_2\le\alpha_{n,r}\|p_n\|_2\\&\le\alpha_{n,r}\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\le\sqrt{\frac{n-r+1}{n+1}}\cdot\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\le\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\le\Psi(t^m).\end{aligned}\tag{24}$$

The definition of the class $W_2^{(r)}(\mathcal{K}_m,\Psi)$ along with (23) and (24) implies that $\mathcal{M}_{n+1}\subset W_2^{(r)}(\mathcal{K}_m,\Psi)$. Then, taking into account the definition of the Bernstein n-width and (18), we obtain
$$\begin{aligned}&\lambda_n\left(W_2^{(r)}(\mathcal{K}_m,\Psi),L_2\right)\ge b_n\left(W_2^{(r)}(\mathcal{K}_m,\Psi),L_2\right)\\&\ge b_n(\mathcal{M}_{n+1};L_2)\ge\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right).\end{aligned}\tag{25}$$

Comparing the upper bound (20) and the lower bound (25), we get the required equality (19). The theorem is proved.

Theorem 2. *Let Ψ be the majorant defining the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$, $m \in \mathbb{N}$, and $r \in \mathbb{R}_+$. Then, for any natural number $n \geq m + r$, we have*

$$\lambda_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right) = E_{n-1}\left(W_2^{(r)}(\mathcal{K}_m, \Psi)\right)$$
$$= \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right), \tag{19}$$

where $\lambda_n(\cdot)$ is any of the n-widths $b_n(\cdot)$, $d_n(\cdot)$, $d^n(\cdot)$, $\delta_n(\cdot)$, and $\Pi_n(\cdot)$.

P r o o f. Let n be a natural number such that $n \geq m + r$. In view of the definition of the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$, relations (15) and (18) imply that

$$\lambda_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right) \leq d_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right)$$
$$\leq E_{n-1}\left(W_2^{(r)}(\mathcal{K}_m, \Psi)\right) \leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right). \tag{20}$$

To find the corresponding lower bound, in view of (18), it suffices to estimate the Bernstein n-width of the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$. On the set $\mathcal{P}_n \cap L_2$, we define the ball

$$\mathcal{M}_{n+1} := \left\{p_n \in \mathcal{P}_n : \|p_n\| \leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)\right\}.$$

Now, we note that, in view of formula (7) and the identity $\alpha_{k,r+m} = \alpha_{k,r}\, \alpha_{k-r,m}$, for an arbitrary $p_n(z) = \sum_{k=0}^{n} a_k(p_n) z^k \in \mathcal{P}_n$, the following equality holds:

$$p_n^{(r+m)}(z) = \sum_{k=r+m}^{n} a_k(p_n) \alpha_{k,r+m} z^{k-r-m} := \sum_{k=r+m}^{n} a_k(p_n) \alpha_{k,r} \cdot \alpha_{k-r,m} z^{k-r-m}.$$

Hence, using the Parseval equality and the inequality $\alpha_{k,r} \leq \alpha_{n,r}, k \leq n$, we obtain the Bernstein type inequality

$$\|p_n^{(r+m)}\| = \left\{\sum_{k=r+m}^{n} |a_k(p_n)|^2 \alpha_{k,r}^2 \cdot \alpha_{k-r,m}^2\right\}^{1/2} \leq \alpha_{n,r} \cdot \alpha_{n-r,m} \|p_n\|. \tag{21}$$

By definition, for the majorant Ψ and for any $0 < \tau_1 \leq \tau_2 \leq 1$, we have the inequality $\tau_1 \Psi(\tau_2) \leq \tau_2 \Psi(\tau_1)$. Therefore, for any $0 < t_1 \leq t_2 \leq 1$, setting $\tau_1 = t_1^m$ and $\tau_2 = t_2^m$, we obtain

$$t_1^{-m} \Psi(t_1^m) \geq t_2^{-m} \Psi(t_2^m). \tag{22}$$

We now show that $\mathcal{M}_{n+1} \subset W_2^{(r)}(\mathcal{K}_m, \Psi)$. Thus, we need to prove that, for any polynomial $p_n \subset \mathcal{M}_{n+1}$,

$$\mathcal{K}_m(p_n^{(r)}, t^m) \leq \Psi(t^m),\ 0 < t \leq 1.$$

Since, by assumption, $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, and $n \geq m + r$, we consider two cases:

$$0 < t \leq \left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)^{1/m}$$

Using the obtained inequality and the second equality in (5), we establish that

$$\sup_{\substack{f\in L_2^{(r)}\\ f\in\mathcal{P}_r}} \frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)},\sqrt{\dfrac{n-r-m+1}{n-r+1}\cdot\dfrac{1}{\alpha_{n-r,m}}}\right)} \geq \frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f_0)}{\mathcal{K}_m\left(f_0^{(r)},\sqrt{\dfrac{n-r-m+1}{n-r+1}\cdot\dfrac{1}{\alpha_{n-r,m}}}\right)} \geq 1. \tag{17}$$

We obtain equality (9) by comparing the upper bound (15) with the lower bound (17). The theorem is proved.

2. Exact values of n-widths of a class of functions

We assume that S is the unit ball in the space L_2, $\Lambda_n\subset L_2$ is an n-dimensional subspace, and $\Lambda^n\subset L_2$ is a subspace of codimension n. Let $\mathcal{L}: L_2\to\Lambda_n$ be a continuous linear operator, let $\mathcal{L}^{\perp}: L_2\to\Lambda_n$ be a continuous linear projection operator, and let $\mathfrak{M}$ be a convex centrally symmetric subset of L_2. The quantities

$$b_n(\mathfrak{M},L_2)=\sup\left\{\sup\left\{\varepsilon>0;\ \varepsilon S\cap\Lambda_{n+1}\subset\mathfrak{M}\right\}:\Lambda_{n+1}\subset L_2\right\},$$

$$d_n(\mathfrak{M},L_2)=\inf\left\{\sup\left\{\inf\left\{\|f-g\|:g\in\Lambda_n\right\}:f\in\mathfrak{M}\right\}:\Lambda_n\subset L_2\right\},$$

$$\delta_n(\mathfrak{M},L_2)=\inf\left\{\inf\left\{\sup\left\{\|f-\mathcal{L}f\|:f\in\mathfrak{M}\right\}:\mathcal{L}L_2\subset\Lambda_n\right\}:\Lambda_n\subset L_2\right\},$$

$$d^n(\mathfrak{M},L_2)=\inf\left\{\sup\left\{\|f\|_{2,\gamma}:f\in\mathfrak{M}\cap\Lambda^n\right\}:\Lambda^n\subset L_2\right\},$$

$$\Pi_n(\mathfrak{M},L_2)=\inf\left\{\inf\left\{\sup\left\{\|f-\mathcal{L}^{\perp}f\|:f\in\mathfrak{M}\right\}:\mathcal{L}^{\perp}L_2\subset\Lambda_n\right\}:\Lambda_n\subset L_2\right\}$$

are called, respectively, the *Bernstein*, *Kolmogorov*, *linear*, *Gelfand*, and *projection* n-widths of the subset $\mathfrak{M}$ in the space L_2. These widths are monotone with respect to n, and the following relation holds (see, for example, [10, 11]):

$$b_n(\mathfrak{M},L_2)\leq d^n(\mathfrak{M},L_2)\leq d_n(\mathfrak{M},L_2)=\delta_n(\mathfrak{M},L_2)=\Pi_n(\mathfrak{M},L_2). \tag{18}$$

We recall (see, for example, [12, p. 25]) that a nondecreasing function Ψ on $\mathbb{R}_+$ is called a k-majorant if the function $t^{-k}\Psi(t)$ is nonincreasing in $\mathbb{R}_+$, $\Psi(0)=0$, and $\Psi(t)\to 0$ as $t\to 0$. For $k=1$, the function Ψ is simply called a majorant.

Let $W_2^{(r)}(\mathcal{K}_m,\Psi)$, $r\in\mathbb{Z}_+, m\in\mathbb{N}$, be the class of all functions $f\in L_2^{(r)}$ whose derivatives $f^{(r)}$ satisfy the condition

$$\mathcal{K}_m(f^{(r)},t^m)\leq\Psi(t^m),\quad 0<t<1.$$

In this definition, Ψ is a majorant, $L_2^{(0)}\equiv L_2$, and $W_2^{(0)}(\mathcal{K}_m,\Psi)=W_2(\mathcal{K}_m,\Psi)$. For any subset $\mathfrak{M}\subset L_2$, we define

$$E_{n-1}(\mathfrak{M})_{L_2}:=\sup\left\{E_{n-1}(f):\ f\in\mathfrak{M}\right\}.$$

We note that, in the Bergman space, values of widths of some classes of analytic functions in a disk were calculated, for example, in [13–19].

where $S_{n-r-1}(g)$ is the partial sum of order $n-r$ of the Fourier series of an arbitrary function $g \in L_2^{(m)}$. In view of (2) and (11), we get

$$\begin{aligned}\|g - S_{n-r-1}(g)\| = E_{n-r-1}(g) &\le \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} E_{n-r-m-1}\left(g^{(m)}\right) \\ &\le \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \left\|g^{(m)}\right\|.\end{aligned} \tag{13}$$

It follows from inequalities (12) and (13) that

$$\begin{aligned}E_{n-1}(f) &\le \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \left\{\|f^{(r)} - g\| + \|g - S_{n-r-1}(g)\|\right\} \\ &\le \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \left\{\|f^{(r)} - g\| + \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \left\|g^{(m)}\right\|\right\}.\end{aligned} \tag{14}$$

Now, we note that the left-hand side of inequality (14) does not depend on $g \in L_2^{(m)}$. Therefore, passing to the infimum over all functions $g \in L_2^{(m)}$ on the right-hand side of (14) and using the definition (8) of $\mathcal{K}$, we get

$$E_{n-1}(f) \le \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \mathcal{K}_m\left(f^{(r)}, \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right).$$

This implies the following upper bound:

$$\sup_{\substack{f \in L_2^{(r)} \\ f \notin \mathcal{P}_r}} \frac{\sqrt{(n+1)/(n-r+1)} \cdot \alpha_{n,r} E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)}, \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)} \le 1, \tag{15}$$

where $\mathcal{P}_r$ is the subspace of complex algebraic polynomials of degree at most r.

To obtain a lower bound of the extremal characteristic on the left-hand side of (15), in (8), we put $f(z) := p_n(z)$, where $p_n(z)$ is an arbitrary complex algebraic polynomial in $\mathcal{P}_n$. Since the function $g(z) \equiv 0$ belongs to the class $L_2^{(m)}$, we obtain from (8) the upper bound

$$\mathcal{K}_m(p_n; t^m)_2 \le \|p_n\|.$$

Since the function $g(z) := p_n(z)$ also belongs to the class $L_2^{(m)}$, we find from (8) that

$$\mathcal{K}_m(p_n; t^m)_2 \le t^m \|p_n^{(m)}\|.$$

Thus, the last two relations imply that, for any element $p_n(z) \in \mathcal{P}_n$,

$$\mathcal{K}_m(p_n; t^m)_2 \le \min\left\{\|p_n\|; t^m \|p_n^{(m)}\|\right\}. \tag{16}$$

We consider the function $f_0(z) = z^n$. Since

$$f_0^{(r+m)} = n(n-1)\cdots(n-r+1)\cdots(n-r-m+1)z^{n-r-m} = \alpha_{n,r} \cdot \alpha_{n-r,m} z^{n-r-m},$$

according to (16), we have

$$\begin{aligned}\mathcal{K}\left(f_0^{(r)}; \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right) &\le \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \|f_0^{(r+m)}\| \\ = \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \cdot \frac{\alpha_{n,r} \cdot \alpha_{n-r,m}}{\sqrt{n-r-m+1}} &= \frac{\alpha_{n,r}}{\sqrt{n-r+1}}.\end{aligned}$$

where

$$\alpha_{k,r} := k(k-1)\cdots(k-r+1), \quad k\in\mathbb{N}, \quad r\in\mathbb{Z}_+, \quad k\geq r.$$

We denote by $L_2^{(r)} := L_2^{(r)}(U)$ $(L_2^{(0)} := L_2(U))$ the class of all functions $f \in L_2$ such that $f^{(r)} \in L_2$ $(r\in\mathbb{Z}_+,\ f^{(0)}\equiv f)$.

1. Sharp estimates of the value of the best approximation by means of $\mathcal{K}$-functionals

In this section, we prove some sharp inequalities relating the value $E_{n-1}(f)$ of the best approximation of functions in the class $L_2^{(r)}$ and Peetre $\mathcal{K}$-functionals. The definition and some properties of Peetre $\mathcal{K}$-functionals are given in [7]. The direct and inverse theorems of the theory of approximation by means of $\mathcal{K}$-functionals were proved in [8, 9]. We define the $\mathcal{K}$-functional constructed by the spaces L_2 and $L_2^{(m)}$ as follows:

$$\mathcal{K}_m(f,t^m)_2 := \mathcal{K}\left(f,t^m;L_2;L_2^{(m)}\right) = \inf\left\{\|f-g\|_2 + t^m\cdot\|g^{(m)}\|_2 : g\in L_2^{(m)}\right\}, \tag{8}$$

where $m\in\mathbb{N}$ and $0<t\leq 1$. We note that a weak equivalence of the $\mathcal{K}$-functional defined by (8) and a special generalized modulus of continuity of order m was established in [8].

Theorem 1. *Let $n,m\in\mathbb{N}$ and $r\in\mathbb{Z}_+$ be arbitrary numbers such that $n\geq r+m$. Then the following equality holds*:

$$\sup_{\substack{f\in L_2^{(r)}\\ f\notin\mathcal{P}_r}} \frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)}, \sqrt{\dfrac{n-r-m+1}{n-r+1}}\cdot\dfrac{1}{\alpha_{n-r,m}}\right)} = 1. \tag{9}$$

P r o o f. Using (7), we easily find that

$$E_{n-r-1}^2(f^{(r)}) = \sum_{k=n}^{\infty}\alpha_{k,r}^2\frac{|c_k(f)|^2}{k-r+1}, \quad r\in\mathbb{Z}_+. \tag{10}$$

Taking into account equality (10), we obtain

$$\begin{aligned} E_{n-1}^2(f) &= \sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1} = \sum_{k=n}^{\infty}\frac{k-r+1}{(k+1)\alpha_{k,r}^2}\cdot\alpha_{k,r}^2\cdot\frac{|c_k(f)|^2}{k-r+1} \\ &\leq \max_{\substack{k\in\mathbb{N}\\ k\geq n}}\left\{\frac{k-r+1}{(k+1)\alpha_{k,r}^2}\right\}\cdot\sum_{k=n}^{\infty}\alpha_{k,r}^2\frac{|c_k(f)|^2}{k-r+1} \\ &= \frac{n-r+1}{n+1}\cdot\frac{1}{\alpha_{n,r}^2}\cdot E_{n-r-1}^2\left(f^{(r)}\right). \end{aligned} \tag{11}$$

Now, for an arbitrary function $f\in L_2^{(r)}$, we write

$$E_{n-1}(f) \leq \sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}E_{n-r-1}\left(f^{(r)}\right) \leq \sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\|f^{(r)}-S_{n-r-1}(g)\|, \tag{12}$$

be the partial sum of order n of the series (2). We form a linear combination of the first n functions of the system $\{\varphi_k(z)\}$:

$$p_{n-1}(z) = \sum_{k=0}^{n-1} d_k \varphi_k(z),$$

where $d_k \in \mathbb{C}$ are arbitrary complex coefficients. We call this linear combination a generalized polynomial. It is well known (see, for example, [1], p.263) that

$$\begin{aligned} E_{n-1}(f) &= \inf\{\|f - p_{n-1}\| : \ d_k \in \mathbb{C}\} \\ &= \|f - S_{n-1}(f)\| = \left\{\sum_{k=n}^{\infty} |a_k(f)|^2\right\}^{1/2}, \end{aligned} \tag{3}$$

where $a_k(f)$ are the Fourier coefficients of the function f defined by (1).

In the case of the mean approximation of complex functions in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ by Fourier series with respect to an orthogonal system of functions $\{\varphi_k(z)\}_{k=0}^{\infty}$ on $\mathcal{D}$, the problem of finding the exact constant in the Jackson-Stechkin inequality was studied in [2]. Recall that Jackson-Stechkin inequalities are inequalities in which the value of the best approximation of a function by a finite dimensional subspace of a given normed space is estimated by the modulus of smoothness of the function itself or some its derivative. In this paper, we use the same methods as in [2, 3, 5, 15].

We study in more detail the case where $\mathcal{D}$ is the unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$. In this case, it is clear that the system of functions $\varphi_k(z) = z^k (k = 0, 1, 2, ...)$ is orthogonal in the disk U:

$$\frac{1}{\pi} \iint_{(U)} \varphi_k(z)\overline{\varphi_l(z)} d\sigma = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^{k+l+1} e^{i(k-l)t} dr dt = 0, \quad k \neq l.$$

However, this system is not orthonormal, since

$$\frac{1}{\pi} \iint_{(U)} |\varphi_k(z)|^2 d\sigma = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^{2k+1} dr dt = \frac{1}{k+1}.$$

Therefore, the system of functions $\varphi_k^*(z) = \sqrt{k+1} z^k$ $(k = 0, 1, 2, ...)$ is orthonormal. We denote by $\mathcal{A}(U)$ the set of all functions f analytic in U. The Maclaurin series of such a function has the form

$$f(z) = \sum_{k=0}^{\infty} c_k(f) z^k, \tag{4}$$

where $c_k(f)$ are the Maclaurin coefficients of f. We note that

$$\|f\|^2 = \sum_{k=0}^{\infty} \frac{|c_k(f)|^2}{k+1}, \quad E_{n-1}^2(f) = \sum_{k=n}^{\infty} \frac{|c_k(f)|^2}{k+1}. \tag{5}$$

It was proved in the monograph [1] that the Fourier series of a function f with respect to the orthonormal system $\varphi_k^*(z) = \sqrt{k+1} z^k$, $k = 0, 1, 2, ...$, coincides with the series (4) for $f \in \mathcal{A}(U)$; i.e.,

$$f(z) = \sum_{k=0}^{\infty} a_k(f) \varphi_k^*(z) = \sum_{k=0}^{\infty} c_k(f) z^k. \tag{6}$$

Therefore, the series (6) can be differentiated term by term any number of times and, according to the Weierstrass theorem [6, p.107], for any $r \in \mathbb{N}$, we get

$$f^{(r)}(z) = \sum_{k=r}^{\infty} c_k(f) k(k-1)\cdots(k-r+1) z^{k-r} := \sum_{k=r}^{\infty} \alpha_{k,r} c_k(f) z^{k-r}, \tag{7}$$

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$\mathcal{K}$-FUNCTIONALS AND EXACT VALUES OF n-WIDTHS IN THE BERGMAN SPACE

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Abstract: In this paper, we consider the problem of mean-square approximation of complex variables functions which are regular in the unit disk of the complex plane. We obtain sharp estimates of the value of the best approximation by algebraic polynomials in terms of $\mathcal{K}$-functionals. Exact values of some widths of the specified class of functions are calculated.

Key words: Bergman space, Best mean-square approximation, $\mathcal{K}$-functional, n-width.

Introduction and preliminary facts

We consider the problem of mean-square approximation by Fourier sums of complex functions f which are regular in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ and belong to the space $L_2 := L_2(\mathcal{D})$ with the finite norm

$$\|f\| := \|f\|_{L_2(\mathcal{D})} = \left(\frac{1}{\pi}\iint_{(\mathcal{D})} |f(z)|^2 d\sigma\right)^{1/2},$$

where the integral is understood in the Lebesgue sense and $d\sigma$ is an element of area.

The study of the mean-square approximation of functions in the domain $\mathcal{D} \subset \mathbb{C}$ is closely related to the theory of orthogonal functions. A sequence of complex functions $\{\varphi_k(z)\}$ $(k = 0, 1, 2, ...)$ is called an orthogonal system on the domain $\mathcal{D}$ if

$$\frac{1}{\pi}\iint_{(\mathcal{D})} \varphi_k(z)\overline{\varphi_l(z)}d\sigma = 0, \qquad k \neq l.$$

Such a sequence of functions is called orthonormal system if

$$\frac{1}{\pi}\iint_{(\mathcal{D})} \varphi_k(z)\overline{\varphi_l(z)}d\sigma = \delta_{k,l},$$

where $\delta_{k,l} = 0,\ \ k \neq l$, and $\delta_{k,k} = 1,\ \ k \in \mathbb{N}$. If $f \in L_2$, then the numbers

$$a_k(f) = \frac{1}{\pi}\iint_{(\mathcal{D})} f(z)\overline{\varphi_k(z)}d\sigma \tag{1}$$

are called the Fourier coefficients of the function f with respect to the orthonormal system $\{\varphi_k(z)\}$ $(k = 0, 1, 2, ...)$. We associate with a given function f its Fourier series with respect to the specified orthogonal system:

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\varphi_k(z). \tag{2}$$

Let

$$S_{n-1}(f, z) = \sum_{k=0}^{n-1} a_k(f)\varphi_k(z)$$

15. **Roman F., Manni C., Speleers H.** Spectral analysis of matrices in Galerkin methods based on generalized B-splines with high smoothness // Numer. Math., 2017. Vol. 135, no. 1. P. 169–216. DOI: 10.1007/s00211-016-0796-z
16. **Mainar E., Peña J.M., Sánchez-Reyes J.** Shape preserving alternatives to the rational Bezier model // Comput. Aided Geom. Design, 2001. Vol. 18. P. 37–60. DOI: 10.1016/S0167-8396(01)00011-5

We now consider separately the case $n = 1$, i.e., the case of $\mathcal{L}$-splines corresponding to the differential operator $\mathcal{L}_4(D) = D^2(D^2 + 1)$. They belong piecewise to the space $span\{1, t, \sin t, \cos t\}$. These splines generalize the well-known cubic splines and have many applications in numerical analysis for the shape preserving approximation, the description of curves and their parametrization, and other problems (see, for instance, [14], [15], [16], and references therein). In particular (see [15]) these splines are attractive from a geometrical point of view, because they are able to provide parameterizations of conic sections with respect to the arc length so that equally spaced points in the parameter domain correspond to equally spaced points on the described curve.

The restriction on the grid step is the least strong here: $h \le \pi/2$, and the "minimal" equidistant grid on the period is $\{0, \pi/2, \pi, 3\pi/2\}$. Theorem 1 gives the existence and uniqueness of spline interpolants for $N \ge 2$. According to Corollary 1, the error of approximation in the class $W_\infty(\mathcal{L}_4)$ is

$$\sup_{f \in W_\infty(\mathcal{L}_4)} \|f - s(f)\|_\infty = \left| 1 + \frac{\pi^2}{8N^2} - \frac{1}{\cos \dfrac{\pi}{2N}} \right|.$$

3. Conclusion

We established that, for 2π-periodic $\mathcal{L}$-splines corresponding to the differential operator (0.1) on the equidistant mesh with the step $h = \pi/N$, the restriction $N > n$ provides the existence and uniqueness of the $\mathcal{L}$-spline interpolant as well as the exact estimates of the error of approximation. This restriction is final, i.e., cannot be replaced by a weaker one.

REFERENCES

1. **Korneichuk N.P.** Splines in approximation theory. Moscow: Nauka, 1984. 352 p. [in Russian]
2. **Micchelli C.A.** Cardinal L-splines // Studies in spline functions and approximation theory, New York etc.: Acad. Press, 1976. P. 203–250.
3. **Novikov S.I.** Approximation of the class $W_\infty^{\mathcal{L}_n}$ by interpolation periodic L-splines // Approximation of functions by polynomials and splines, Sverdlovsk: Akad. Nauk SSSR, Ural. Sc. Center, 1985. P. 118–126. [in Russian]
4. **Novikov S.I.** Generalization of the Rolle theorem // East J. Approx., 1995. Vol. 1, no. 4. P. 571–575.
5. **Nguen Thi T.H.** The operator $D(D^2 + 1^2) \cdots (D^2 + n^2)$ and trigonometric interpolation // Anal. Math., 1989. Vol. 15, no. 4. P. 291–306. [in Russian]
6. **Nguen Thi T.H.** Extremal problems for some classes of smooth periodic functions // Doctoral dissertation, 1994. Moscow: Steklov Institute of Math. 219 p. [in Russian]
7. **Prudnikov A.P., Brychkov Yu.A., Marichev O.I.** Integrals and series. Elementary functions, Moscow: Nauka, 1981. 800 p. [in Russian]
8. **Schoenberg I.J.** On Micchelli's theory of cardinal L-splines // Studies in spline functions and approximation theory, New York etc.: Acad. Press, 1976. P. 251–276.
9. **Shevaldin V.T.** A problem of extremal interpolation // Mat. Zametki, 1981. Vol. 29, no. 4. P. 603-622. [in Russian]
10. **Shevaldin V.T.** Interpolation periodic L-splines with uniform nodes // Approximation of functions by polynomials and splines, Sverdlovsk: Akad. Nauk SSSR, Ural. Sc. Center, 1985. P. 140–147. [in Russian]
11. **Stechkin S.B., Subbotin Yu.N.** Splines in numerical mathematics. Moscow: Nauka, 1976. 248 p. [in Russian]
12. **Tikhomirov V.M.** Best methods of approximation and interpolation of differentiable functions in the space $C[-1, 1]$ // Mat. Sb., 1969. Vol. 80, no. 122. P. 290–304. [in Russian]
13. **Zhensykbaev A.A.** Approximation of differentiable periodic functions by splines on a uniform subdivision // Mat. Zametki, 1973. Vol. 13, no. 6. P. 807–816. [in Russian]
14. **Zhang J.** C-curves: an extension of cubic curves // Comput. Aided Geom. Design, 1996. Vol. 13. P. 199–217.

The inequality $N > n$ cannot be replaced by a weaker one. Indeed, if $N = n$, then the function $\sin nx$ interpolates the sequence $y \equiv 0$ at the points of Δ_N. This function lies in the kernel of the linear differential operator (0.1) and can be interpreted as an element of the space $S(\mathcal{L}_{2n+2}, \Delta_N)$. Theorem 1 is proved. $\square$

P r o o f of Theorem 2 is based on the ideas of [12]. Let $N > n$. Suppose that (0.3) fails; i.e., there exist a point $x_* \in [0, 2\pi)$ and a function $f \in W_\infty(\mathcal{L}_{2n+2})$ such that the inequality

$$|f(x_*) - s(f)(x_*)| > 2|A_n(x_*)|$$

holds. Define $\delta(x) = f(x) - s(f)(x)$. This function is zero at the points of the mesh Δ_N. According to Lemma 3, the function $A_n(x)$ vanishes at the same points. From these facts, we have $x_* \notin \Delta_N$. Therefore, there is a number λ, $0 < |\lambda| < 1$, such that $\lambda\delta(x_*) = 2A_n(x_*)$.

We now introduce the function $\Delta(x) = \lambda\delta(x) - 2A_n(x)$. It is zero at all points of the set $\Delta_N \cup \{x_*\}$ and possibly also at some other points. Therefore $Z(\Delta(x), \mathbb{T}) \geq 2N + 1$. It is clear that $\Delta \in G_{2n+1}(\mathbb{T}) \backslash \mathcal{T}_n$. We denote $\mathcal{L}_{2n+1}(D) = D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2)$, apply Lemma 4, and obtain

$$\nu\left(\mathcal{L}_{2n+1}(D)\Delta(x), \mathbb{T}\right) \geq 2N + 1. \tag{2.1}$$

From (0.1) and the definition of almost trigonometric splines, we have the equalities $\mathcal{L}_{2n+1}(D)\delta(x) = \mathcal{L}_{2n+1}(D)f(x) - c_j$ on every interval $[jh, (j+1)h)$, $j = 0, 1, \ldots, 2N - 1$, where c_j are constants. Using the Lagrange finite increments formula and the inequality $|\lambda| < 1$, we obtain

$$|\mathcal{L}_{2n+1}(D)(\lambda\delta(t')) - \mathcal{L}_{2n+1}(D)(\lambda\delta(t''))| < |\mathcal{L}_{2n+1}(D)f(t') - \mathcal{L}_{2n+1}(D)f(t'')|$$

$$= |\mathcal{L}_{2n+2}(D)f(\xi)| \cdot |t' - t''| \ \leq |t' - t''|$$

on an arbitrary subinterval $[t', t''] \subset [jh, (j+1)h)$ for every interpolated function of our class. From (0.2), it follows that $\mathcal{L}_{2n+1}(D)(2A_n(x)) = x - h/2$ $\forall x \in [0, h)$. Hence, $|t' - t''| = |\mathcal{L}_{2n+1}(D)(2A_n(t')) - \mathcal{L}_{2n+1}(D)(2A_n(t''))|$. Thus,

$$|\mathcal{L}_{2n+1}(D)(\lambda\delta(t')) - \mathcal{L}_{2n+1}(D)(\lambda\delta(t''))| < |\mathcal{L}_{2n+1}(D)(2A_n(t')) - \mathcal{L}_{2n+1}(D)(2A_n(t''))|.$$

It is easy to see that if $|a| < |b|$, then $\operatorname{sign}(b - a) = \operatorname{sign} b$. Applying this fact, we come to the conclusion that the function $\mathcal{L}_{2n+1}(D)\Delta(x)$ changes sign no more than once in every interval $[jh, (j+1)h)$. If $\mathcal{L}_{2n+1}(D)\Delta(x)$ changes sign at the point jh (this is possible if the function is discontinuous at jh), then $\mathcal{L}_{2n+1}(D)\Delta(x)$ preserves sign in one of two adjacent intervals $((j-1)h, jh)$ or $(jh, (j+1)h)$. Thus, we arrive at the inequality

$$\nu\left(\mathcal{L}_{2n+1}(D)\Delta(x), \mathbb{T}\right) \leq 2N.$$

The obtained inequality contradicts to (2.1). The simple observation that inequality (0.3) turns into an equality for $f = 2A_n$ completes the proof. $\square$

Corollary 1. *If $N > n$, then*

$$\sup_{f \in W_\infty(\mathcal{L}_{2n+2})} \|f - s(f)\|_p = 2\|A_n\|_p, \quad 1 \leq p < \infty,$$

and

$$\sup_{f \in W_\infty(\mathcal{L}_{2n+2})} \|f - s(f)\|_\infty = \left| \frac{h^2}{8(n!)^2} + 4\sum_{\nu=1}^{n} \frac{(-1)^\nu \sin^2 \dfrac{\nu h}{4}}{\nu^2 \ (n-\nu)! \ (n+\nu)! \ \cos \dfrac{\nu h}{2}} \right|.$$

Lemma 3. *If $N > n$, then $x = 0$ is the unique zero of $A_n(x)$ in $[0, h)$ and this zero is simple.*

P r o o f. By Lemma 1, $A_n(0) = 0$. Moreover, the function $A_n(x)$ coincides, up to a nonzero constant, with some function $P_n(x)$ introduced in [9]. It was proved in [9] that if $N > n$, then $P_n(x)$ has a unique zero in $[0, h)$ and this zero is simple. Therefore, $A_n(x)$ has the same property. □

To prove our two theorems, we also need the periodic analog of the Rolle theorem on the relation between the number of zeros of a smooth function $\varphi(x)$ and the number of sign changes of $D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)\varphi(x)$ on $\mathbb{T}$.

We say that a continuous function f changes sign at some point t_0 if the inequality $f(t_0-\varepsilon)f(t_0+\varepsilon) < 0$ holds for all sufficiently small $\varepsilon > 0$. If f has a jump at the point t_0, then, instead of $f(t_0-\varepsilon)$ and $f(t_0+\varepsilon)$, we write $\lim\limits_{t\to t_0-0} f(t)$ and $\lim\limits_{t\to t_0+0} f(t)$, respectively. Denote by $Z(f, \mathbb{T})$ the number of zeros of the function f on $\mathbb{T}$, and by $\nu(f, \mathbb{T})$ the number of sign changes of $f \not\equiv 0$ on $\mathbb{T}$ (the number of sign changes of $f \equiv 0$ is not defined). The number $\nu(f, \mathbb{T})$ is always even. We denote by $G(\mathbb{T})$ the set of 2π-functions of bounded variation with a finite number of jumps on the period and absolutely continuous on all intervals of continuity. We also denote by $G_m(\mathbb{T})$ the set of 2π-periodic functions whose derivatives of order $m-2$ are absolutely continuous on $\mathbb{T}$ and $f^{(m-1)} \in G(\mathbb{T})$. Let $\mathcal{T}_n$ be the set of trigonometric polynomials of order at most n.

Lemma 4. *For every function $f \in G_{2n+1}(\mathbb{T})\backslash\mathcal{T}_n$, the following inequality holds*:

$$\nu\big(D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)f, \mathbb{T}\big) \geq Z(f, \mathbb{T}).$$

This result was established by Nguen [5] (see also [6]) and is the periodic analog of the Rolle theorem for the trigonometric differential operator.

Note that the periodic analog of the Rolle theorem in the form of Lemma 4 exists not for any linear differential operator. More detailed information on some results and open problems in this area can be found in [4] and references therein.

2. Proofs of Theorems

We now pass directly to the proofs of Theorems 1 and 2.

P r o o f of Theorem 1. Let $N > n$. We prove that if $s \in S(\mathcal{L}_{2n+2}, \Delta_N)$ and $s(jh) = 0$ $\forall j \in \mathbb{Z}$, then $s \equiv 0$. After this, the existence and uniqueness of the interpolating periodic almost trigonometric spline for every interpolated periodic sequence is a simple consequence of the Kramer theorem for the corresponding system of linear algebraic equations.

Suppose that there exist $s_1, s_2 \in S(\mathcal{L}_{2n+2}, \Delta_N)$ such that $s_k(jh) = 0$ $\forall j \in \mathbb{Z}$ $(k = 1, 2)$ and $s_1 \not\equiv s_2$. This means that there is a point $x_* \notin \Delta_N$ such that $s_1(x_*) \neq s_2(x_*)$. Let $s_1(x_*) \neq 0$ and $C = s_2(x_*)/s_1(x_*)$. Then the function $\varphi(x) = Cs_1(x) - s_2(x)$ has the following properties:

1) $\varphi \in S(\mathcal{L}_{2n+2}, \Delta_N)$;
2) $\varphi(jh) = 0, \quad j = 0, 1, \ldots, 2N-1$;
3) $\varphi(x_*) = 0$.

Thus, $\varphi(x)$ has at least $2N+1$ zeros on the period. From Lemma 4, we have

$$\nu(D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)\varphi, \mathbb{T}) \geq 2N+1.$$

But $D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)\varphi(x)$ is a piecewise constant function with possible jumps at the points of the mesh Δ_N. Therefore, this function cannot change sign more than $2N$ times on $\mathbb{T}$. We have a contradiction from which it easily follows that $s_1 \equiv s_2 \equiv 0$.

P r o o f. By easy calculations, we verify that $A_n(h) = A_n(0) = 0$ and $A'_n(x)\mid_{x=h} = -A'_n(x)\mid_{x=0}$. Further,

$$A''_n(x)\mid_{x=h} = A''_n(x)\mid_{x=0} = \frac{1}{2(n!)^2} - \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{(n-\nu)!\,(n+\nu)!}.$$

Using the known identity [7, Ch.IV, § 4.2.1, eq. 4], we obtain

$$\sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{(n-\nu)!\,(n+\nu)!} = \sum_{m=0}^{n-1} \frac{(-1)^{m+n-1}}{m!\,(2n-m)!} = \frac{(-1)^{n-1}}{(2n)!} \sum_{m=0}^{n-1} (-1)^m \binom{2n}{m} = \frac{1}{2(n!)^2}.$$

From this, it follows that $A''_n(x)\mid_{x=h} = A''_n(x)\mid_{x=0} = 0$.

For $j = 3, 4, \ldots, 2n+1$, we have

$$A_n^{(j)}(x) = \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}\nu^{j-2}\cos(\nu(x-h/2)+\pi j/2)}{(n-\nu)!\,(n+\nu)!\,\cos(\nu h/2)}.$$

For $j = 2k+1$ $(k = 1, 2, \ldots, n)$, easy calculations yield

$$A_n^{(2k+1)}(x)\mid_{x=h} = -A_n^{(2k+1)}(x)\mid_{x=0} = (-1)^k \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}\,\nu^{2k-1}\,\tan(\nu h/2)}{(n-\nu)!\,(n+\nu)!}.$$

For $j = 2k$ $(k = 2, 3, \ldots, n)$, we obtain

$$A_n^{(2k)}(x)\mid_{x=h} = A_n^{(2k)}(x)\mid_{x=0} = (-1)^k \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}\,\nu^{2k-2}}{(n-\nu)!\,(n+\nu)!}$$

$$= \frac{(-1)^{n+k}}{(2n)!} \sum_{m=0}^{n-1} (-1)^m (n-m)^{2k-2} \binom{2n}{m} = 0.$$

Here, we used the identity [7, Ch.IV, § 4.2.2, eq. 34]. The lemma is proved. □

We now extend the function $A_n(x)$ from $[0, h]$ to the whole real line by setting $A_n(x+h) = -A_n(x)$. Lemma 1 gives that A_n belongs to $C^{(2n+1)}(\mathbb{R})$ and is 2π-periodic.

Lemma 2. *If $N > n$, then $\mathcal{L}_{2n+2}(D)(2A_n(x)) = \operatorname{sign}\sin Nx,\ x \in \mathbb{R}$.*

P r o o f. Let $0 \le x \le h$. Since

$$\sin\frac{\nu x}{2}\,\sin\frac{\nu(x-h)}{2} = A_\nu \cos\nu x + B_\nu \sin\nu x + C_\nu, \quad \nu = 1, 2, \ldots, n,$$

where A_ν, B_ν and C_ν are independent of x, the sum on the right-hand side of (0.2) vanishes by the differential operator $D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)$. Taking into account that the factors on the right-hand side of (0.1) can be rearranged, we obtain

$$D^2(D^2+1^2)\cdots(D^2+n^2)\left(\frac{x(x-h)}{2(n!)^2}\right) = \frac{1}{2(n!)^2}(D^2+1^2)\cdots(D^2+n^2)(x(x-h))'' = 1.$$

For $x \in \mathbb{R}\backslash[0, h]$, we use the equality $A_n(x+h) = -A_n(x)$. □

The next statement is a special case of a result proved in [9] for an arbitrary linear differential operator with constant real coefficients.

constant. This term is not standard, and we use it only not to specify every time by what differential operator the considered splines are defined.

We interpolate at the knots of the mesh Δ_N by elements from $S(\mathcal{L}_{2n+2}, \Delta_N)$; i.e., for every bounded $2N$-periodic sequence $y = \{y_\nu : \nu \in \mathbb{Z}\}$, $y_\nu = y_{\nu+2N}$, we consider the interpolation problem: to find $s \in S(\mathcal{L}_{2n+2}, \Delta_N)$ such that $s(\nu h) = y_\nu, \nu \in \mathbb{Z}$.

For interpolation by polynomial splines, the existence, uniqueness and estimates of the error of approximation in many classes of functions are well-known (see, for instance, [1, Ch. V], [11], [12], [13], and references therein).

The existence and uniqueness of periodic interpolating L-splines corresponding to an arbitrary linear differential operator with constant real coefficients were established in [10]. As far as almost trigonometric splines are concerned the result in [10] means that if $N > n$, then for every bounded $2N$-periodic interpolated sequence, there exists a unique interpolating almost trigonometric spline.

In the present paper, we give another proof of this result and observe such an important feature that the inequality $N > n$ cannot be replaced by a weaker one (Theorem 1). After this, for $N > n$, we obtain a sharp estimate of the error of pointwise approximation by periodic interpolating almost trigonometric splines in the class of functions $W_\infty(\mathcal{L}_{2n+2})$ (Theorem 2).

Theorem 1. *If $N > n$, then, for every bounded $2N$-periodic sequence $\{y_\nu\}_{\nu\in\mathbb{Z}}$, $y_\nu = y_{\nu+2N}$, there exists a unique $s \in S(\mathcal{L}_{2n+2}, \Delta_N)$ such that $s(\nu h) = y_\nu, \nu \in \mathbb{Z}$.*

If $N \le n$, then periodic interpolating almost trigonometric spline cannot exist.

Let $N > n$. We set

$$A_n(x) = \frac{x(x-h)}{4(n!)^2} + 2\sum_{\nu=1}^{n} \frac{(-1)^\nu \sin\dfrac{\nu x}{2}\ \sin\dfrac{\nu(x-h)}{2}}{\nu^2\ (n-\nu)!\ (n+\nu)!\ \cos\dfrac{\nu h}{2}} \tag{0.2}$$

for $0 \le x \le h$ and extend $A_n(x)$ to the whole real line by the equality $A_n(x+h) = -A_n(x)$ for $x \in \mathbb{R}\backslash[0, h]$.

We show that $A_n \in C^{(2n+1)}(\mathbb{T})$. In the class $W_\infty(\mathcal{L}_{2n+2})$, the deviation from the periodic interpolating almost trigonometric splines is estimated by this function.

Theorem 2. *If $N > n$, then, for every function $f \in W_\infty(\mathcal{L}_{2n+2})$, the inequality*

$$|f(x) - s(f)(x)| \le 2|A_n(x)| \tag{0.3}$$

holds at any point $x \in \mathbb{R}$. The inequality turns into an equality for $f(x) = 2A_n(x)$.

For interpolation by periodic polynomial splines, inequality (0.3) was proved by Tikhomirov [12]. For $N > 3^{n-1}n$, inequality (0.3) is a particular case of the author's result [3]. For periodic trigonometric splines, the corresponding result was established by Nguen [5], [6, Ch. 2, §6].

1. Auxiliary results

First, we study the properties of the function $A_n(x)$.

Lemma 1. *If $N > n$, then*

$$A_n^{(j)}(x)\,|_{x=h} = -A_n^{(j)}(x)\,|_{x=0}, \quad j = 1, 3, \ldots, 2n+1,$$

and

$$A_n^{(j)}(x)\,|_{x=h} = A_n^{(j)}(x)\,|_{x=0} = 0, \quad j = 0, 2, \ldots, 2n.$$

ON INTERPOLATION BY ALMOST TRIGONOMETRIC SPLINES[1]

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Abstract: The existence and uniqueness of an interpolating periodic spline defined on an equidistant mesh by the linear differential operator $\mathcal{L}_{2n+2}(D) = D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)$ with $n \in \mathbb{N}$ are reproved under the final restriction on the step of the mesh. Under the same restriction, sharp estimates of the error of approximation by such interpolating periodic splines are obtained.

Key words: Splines, Interpolation, Approximation, Linear differential operator.

Introduction

Let $D = d/dx$, $n \in \mathbb{N}$, and let

$$\mathcal{L}_{2n+2}(D) = D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2) \tag{0.1}$$

be the $(2n+2)$th-order linear differential operator with constant real coefficients. We denote the characteristic polynomial of $\mathcal{L}_{2n+2}(D)$ by p_{2n+2}, and let $T_{2n+2} = \{0, 0, \pm i, \ldots, \pm in\}$ be the set of its zeros, with each zero repeated according to its multiplicity, where i is the imaginary unit. The kernel of the differential operator (0.1) is the linear space spanned by the system of functions $\{1, x, \sin x, \cos x, \ldots, \sin nx, \cos nx\}$.

Denote by $\mathbb{T}$ the circumference considered as the interval $[0, 2\pi]$ with identified ends, and let $\|\cdot\|_{L_p(\mathbb{T})} = \|\cdot\|_p$ $(1 \le p \le \infty)$ with the usual modification in the case $p = \infty$.

We associate with the differential operator $\mathcal{L}_{2n+2}(D)$ the standard class of differentiable functions

$$W_\infty(\mathcal{L}_{2n+2}) = \{f \in C^{(2n+1)}(\mathbb{T}) : \ f^{(2n+1)} \ \text{ is abs. cont.}, \ \|\mathcal{L}_{2n+2}(D)f\|_\infty \le 1\}.$$

Let $N \in \mathbb{N}$ and $h = \pi/N$. Denote by $\Delta_N = \{jh : j = 0, 1, \ldots, 2N-1\}$ the uniform mesh on $[0, 2\pi)$ which can be extended on $\mathbb{R}$ if required; h is the step of the mesh.

We say that a 2π-periodic function s_{2n+2} is a periodic *almost trigonometric spline* with knots at the points of Δ_N if s_{2n+2} satisfies the following conditions:

1) $s_{2n+2} \in C^{(2n)}(\mathbb{T})$,

2) $\mathcal{L}_{2n+2}(D)s_{2n+2}(x) = 0 \quad \forall x \in (jh, (j+1)h), \quad j \in \mathbb{Z}$.

The set of all almost trigonometric splines is denoted by $S(\mathcal{L}_{2n+2}, \Delta_N)$.

Almost trigonometric splines are a special case of the large family of $\mathcal{L}$-splines defined by linear differential operators (see [2], [3], [8], and others).

The term “almost trigonometric spline” is explained by the fact that such a spline is formed by functions which differ from trigonometric polynomials for only one addend ax, where a is some

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8. **Hruslov E.Ya.** The asymptotic behavior of solutions of the second boundary value problem under fragmentation of the boundary of the domain // Math. USSR-Sb., 1979. Vol. 35, no. 2. P. 266–282. DOI: 10.1070/SM1979v035n02ABEH001474
9. **Kovalevsky A.A.** On the convergence of solutions of variational problems with bilateral obstacles in variable domains // Proc. Steklov Inst. Math. (Suppl.), 2017. Vol. 296, suppl. 1. P. 151–163. DOI: 10.1134/S0081543817020146
10. **Kovalevsky A.A.** On the convergence of solutions to bilateral problems with the zero lower constraint and an arbitrary upper constraint in variable domains // Nonlinear Anal., 2016. Vol. 147. P. 63–79. DOI: 10.1016/j.na.2016.09.001
11. **Vainberg M.M.** Variational method and method of monotone operators in the theory of nonlinear equations. New York: Wiley, 1974. 356 p.
12. **Kovalevskii A.A.** Some problems connected with the problem of averaging variational problems for functionals with a variable domain // Current Analysis and its Applications. Kiev: Naukova dumka, 1989. P. 62–70.
13. **Kovalevsky A.A.** Obstacle problems in variable domains // Complex Var. Elliptic Equ., 2011. Vol. 56, no. 12. P. 1071–1083. DOI: 10.1080/17476933.2010.504842
14. **Adams R.A.** Sobolev spaces. New York: Academic Press, 1975. 286 p.
15. **Malý J., Ziemer W.P.** Fine regularity of solutions of elliptic partial differential equations. Providence: AMS, 1997. 291 p.
16. **Kovalevskii A.A.** G-convergence and homogenization of nonlinear elliptic operators in divergence form with variable domain // Russian Acad. Sci. Izv. Math., 1995. Vol. 44, no. 3. P. 431–460. DOI: 10.1070/IM1995v044n03ABEH001607
17. **Kovalevskij A.A.** Conditions of the Γ-convergence and homogenization of integral functionals with different domains of the definition // Dokl. Akad. Nauk Ukr. SSR, 1991. No. 4. P. 5–8. [in Russian]
18. **Kovalevskii A.A.** On necessary and sufficient conditions for the Γ-convergence of integral functionals with different domains of definition // Nelinein. Granichnye Zadachi, 1992. Vol. 4. P. 29–39. [in Russian]
19. **Kovalevskii A.A., Rudakova O.A.** On the Γ-compactness of integral functionals with degenerate integrands // Nelinein. Granichnye Zadachi, 2005. Vol. 15. P. 149–145. [in Russian]
20. **Rudakova O.A.** On Γ-convergence of integral functionals defined on various weighted Sobolev spaces // Ukrainian Math. J., 2009. Vol. 61. P. 121–139. DOI: 10.1007/s11253-009-0193-1
21. **Kovalevsky A.A.** On L^1-functions with a very singular behaviour // Nonlinear Anal., 2013. Vol. 85. P. 66–77. DOI: 10.1016/j.na.2013.02.017
22. **Murat F.** Sur l'homogeneisation d'inequations elliptiques du 2ème ordre, relatives au convexe $K(\psi_1, \psi_2) = \{v \in H_0^1(\Omega) \,|\, \psi_1 \leqslant v \leqslant \psi_2$ p.p. dans $\Omega\}$. 1976. Publ. Laboratoire d'Analyse Numérique, no. 76013. Univ. Paris VI. 23 p.
23. **Kovalevsky A.A.** Variational problems with unilateral pointwise functional constraints in variable domains // Trudy Inst. Mat. i Mekh. UrO RAN, 2017. Vol. 23, no. 2. P. 133–150. [in Russian] DOI: 10.21538/0134-4889-2017-23-2-133-150
24. **Kovalevsky A.A., Rudakova O.A.** Variational problems with pointwise constraints and degeneration in variable domains // Differ. Equ. Appl., 2009. Vol. 1, no. 4. P. 517–559. DOI: 10.7153/dea-01-29.
25. **Gilbarg D., Trudinger N.S.** Elliptic partial differential equations of second order. Berlin: Springer, 1983. 518 p. DOI: 10.1007/978-3-642-96379-7
26. **Kuratowski K.** Topology. Vol. I. New York: Academic Press, 1966. 580 p.
27. **Mosco U.** Convergence of convex sets and of solutions of variational inequalities // Adv. Math., 1969. Vol. 3, no. 4. P. 510–585.

4. Conclusion

In this paper, we have formulated and have discussed some results on the convergence of sequences of minimizers and minimum values of functionals $F_s + G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ on sets of functions defined by bilateral constraints in domains Ω_s. These domains are assumed to be contained in a bounded domain Ω of $\mathbb{R}^n$. The functionals F_s are integral and convex, and their integrands satisfy the bilateral estimate $c_1|\xi|^p - \mu_s(x) \leqslant f_s(x,\xi) \leqslant c_2|\xi|^p + \mu_s(x)$ for almost every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$, where c_1 and c_2 are positive constants and μ_s are nonnegative functions such that the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded. The functionals G_s are assumed to be weakly continuous on the corresponding Sobolev spaces. They are generally not integral and play a subordinate role.

We have considered two cases: the case of regular constraints, i.e., constraints lying in the Sobolev space $W^{1,p}(\Omega)$, and the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative function. In both cases, a certain connection of the spaces $W^{1,p}(\Omega_s)$ with the space $W^{1,p}(\Omega)$, the Γ-convergence of the functionals F_s, and a convergence of the functionals G_s are essentially used. At the same time, each of these cases has a distinctive feature. In the first case, it is required that the difference between the upper and lower constraints be positive almost everywhere. In the second case, this requirement is absent. However, in the latter case, it is assumed that $\|\mu_s\|_{L^1(\Omega_s)} \to 0$ and it is required that the exhaustion condition of the domain Ω by the domains Ω_s be satisfied.

We have given a series of results involving the exhaustion condition. In particular, we have obtained an equivalent statement of this condition and, using it, have proved the $\mathcal{H}$-convergence of sets of functions defined by bilateral (generally irregular) constraints in the domains Ω_s.

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REFERENCES

1. **De Giorgi E., Franzoni T.** Su un tipo di convergenza variazionale // Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 1975. Vol. 58, no. 6. P. 842–850.
2. **Zhikov V.V.** Questions of convergence, duality, and averaging for functionals of the calculus of variations // Math. USSR-Izv., 1984. Vol. 23, no. 2. P. 243–276. DOI: 10.1070/IM1984v023n02ABEH001466
3. **Dal Maso G.** An introduction to Γ-convergence. Boston: Birkhäuser, 1993. 352 p. DOI: 10.1007/978-1-4612-0327-8
4. **Zhikov V.V.** On passage to the limit in nonlinear variational problems // Russian Acad. Sci. Sb. Math., 1993. Vol. 76, no. 2. P. 427–459. DOI: 10.1070/SM1993v076n02ABEH003421
5. **Kovalevskii A.A.** Averaging of variable variational problems // Dokl. Akad. Nauk Ukr. SSR. Ser. A, 1988. No. 8. P. 6–9. [in Russian]
6. **Kovalevskii A.A.** On the connectedness of subsets of Sobolev spaces and the Γ-convergence of functionals with varying domain of definition // Nelinein. Granichnye Zadachi, 1989. Vol. 1. P. 48–54. [in Russian]
7. **Kovalevskii A.A.** On the Γ-convergence of integral functionals defined on Sobolev weakly connected spaces // Ukrainian Math. J., 1996. Vol. 48, no. 5. P. 683–698. DOI: 10.1007/BF02384235

We now conclude that the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set U. $\square$

Remark 2. Concerning some notions of convergence of sets lying in the same space, see, for instance, [26, 27]. Our notion of $\mathcal{H}$-convergence of sets lying generally in variable spaces differs from the notions of convergence of sets in the sense of Kuratowski [26, Section 29] and in the sense of Mosco [27, Definition 1.1] even in the case of sets belonging to the same space.

We give one more result involving condition $(*')$ of Theorem 2.

Proposition 10. *Let conditions $(*_1)$, $(*_2)$, $(*_4)$, and $(*_5)$ of Theorem 1 be satisfied. In addition, let condition $(*')$ of Theorem 2 be satisfied. Then there exist positive constants b_1 and b_2 such that, for every function $v \in W^{1,p}(\Omega)$, we have $(F+G)(v) \geqslant b_1\|v\|^p_{W^{1,p}(\Omega)} - b_2$.*

P r o o f. By condition $(*_2)$ of Theorem 1, there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in Ω_s. We set $\lambda = \sup\limits_{s\in\mathbb{N}} \|l_s\|$. It is not difficult to find that λ is a real number such that $\lambda \geqslant 1$. Next, let $v \in W^{1,p}(\Omega)$. By virtue of condition $(*_4)$ of Theorem 1, there exists a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $F_s(w_s) \to F(v)$. The first of these limit relations and condition $(*_5)$ of Theorem 1 imply that $G_s(w_s) \to G(v)$. Thus,

$$(F_s + G_s)(w_s) \to (F+G)(v). \tag{27}$$

In view of (7), we have

$$\forall s \in \mathbb{N}, \qquad (F_s + G_s)(w_s) \geqslant c_5\|w_s\|^p_{W^{1,p}(\Omega_s)} - c_6. \tag{28}$$

This along with (27) implies that the sequence of norms $\|w_s\|_{W^{1,p}(\Omega_s)}$ is bounded. Now, since condition $(*_1)$ of Theorem 1 and condition $(*')$ of Theorem 2 are satisfied, we deduce from Proposition 5 that $l_s w_s \to v$ weakly in $W^{1,p}(\Omega)$. Therefore,

$$\liminf_{s\to\infty} \|l_s w_s\|_{W^{1,p}(\Omega)} \geqslant \|v\|_{W^{1,p}(\Omega)}. \tag{29}$$

Moreover, we have

$$\forall s \in \mathbb{N}, \qquad \|l_s w_s\|_{W^{1,p}(\Omega)} \leqslant \lambda \|w_s\|_{W^{1,p}(\Omega_s)}. \tag{30}$$

From (27)–(30), we derive that $(F+G)(v) \geqslant c_5\lambda^{-p}\|v\|^p_{W^{1,p}(\Omega)} - c_6$. $\square$

We observe that condition $(*')$ of Theorem 2 is essential for the conclusion of Proposition 10. In this regard, see [10, Example 4.3].

We complete the exposition of the results related to condition $(*')$ of Theorem 2 with the following proposition.

Proposition 11. *Assume that $c > 0$ and, for every open set H of $\mathbb{R}^n$ such that $H \subset \Omega$, we have $\liminf\limits_{s\to\infty} \operatorname{meas}(H \cap \Omega_s) \geqslant c \operatorname{meas} H$. Then condition $(*')$ of Theorem 2 is satisfied.*

Concerning the proof of this result, see, for instance, [10]. We also remark that the condition of Proposition 11 is satisfied in the case where the domains Ω_s have a perforated structure of the same kind as the structure of the domains considered in [16, Section 2].

Finally, we note that condition $(*'')$ of Theorem 2 is also important for the conclusion of this theorem. In this regard, see [10, Example 4.4]. Obviously, condition $(*'')$ of Theorem 2 is satisfied if all the functions μ_s are zero in the corresponding domains or if, for instance, for every $s \in \mathbb{N}$, we have $\mu_s = \alpha_s \mu|_{\Omega_s}$, where $\{\alpha_s\} \subset [0, +\infty)$, $\alpha_s \to 0$, and μ is a nonnegative function in $L^1(\Omega)$.

Obviously, $\varphi \leqslant \psi$ in Ω. Let, for every $s \in \mathbb{N}$, $U_s = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}$, and let $U = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\}$. Clearly, the set U is nonempty. Thus, all the conditions of Proposition 8 are satisfied except for condition $(*')$ of Theorem 2. At the same time, the sequence $\{U_s\}$ does not $\mathcal{H}$-converge to the set U. In fact, suppose that the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set U. Then, taking the sequence $v_s \in W^{1,p}(\Omega_s)$ such that, for every $s \in \mathbb{N}$, $v_s = 1$ in Ω_s, we find that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in U$ such that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$. Hence, $v = 1$ a.e. in $\Omega \setminus B$. Therefore, $v - 1 \in \overset{\circ}{W}{}^{1,p}(\Omega)$. Moreover, since $v \in U$, we have $v = 0$ a.e. in B. Thus, $|\nabla v| = 0$ a.e. in Ω. Then, fixing a number r such that $1 < r < \min\{p, n\}$ and taking into account that $v - 1 \in \overset{\circ}{W}{}^{1,r}(\Omega)$, we apply the corresponding Sobolev inequality for the function $v - 1$ and find that $v = 1$ a.e. in Ω. However, this contradicts the fact that $v = 0$ a.e. in B. The obtained contradiction proves that the sequence $\{U_s\}$ does not $\mathcal{H}$-converge to the set U.

Although, in the general case, condition $(*')$ of Theorem 2 is essential for the $\mathcal{H}$-convergence of sets defined by bilateral constraints, in the case of regular constraints, this condition does not play any role for the $\mathcal{H}$-convergence of the corresponding sets. We demonstrate this by proving the following result.

Proposition 9. *Assume that conditions $(*_1)$ and $(*_2)$ of Theorem* 1 *are satisfied. Let $\varphi, \psi \in W^{1,p}(\Omega)$, and let $\varphi \leqslant \psi$ a.e. in Ω. Let, for every $s \in \mathbb{N}$, $U_s = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi$ a.e. in $\Omega_s\}$, and let $U = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi$ a.e. in $\Omega\}$. Then the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set U.*

P r o o f. As in the proof of Proposition 8, we establish that, for every function $v \in U$, there exists a sequence $w_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$.

Next, we fix an arbitrary sequence $v_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$. In view of condition $(*_2)$ of Theorem 1, there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and

$$\forall s \in \mathbb{N}, \quad q_s(l_s v_s) = v_s \text{ a.e. in } \Omega_s. \tag{24}$$

It is easy to see that the sequence $\{l_s v_s\}$ is bounded in $W^{1,p}(\Omega)$. For every $s \in \mathbb{N}$, we set

$$z_s = \min\{\max\{l_s v_s, \varphi\}, \psi\}.$$

We have $\{z_s\} \subset U$ and the sequence $\{z_s\}$ is bounded in $W^{1,p}(\Omega)$. Moreover, using (24) and the inclusions $v_s \in U_s$, we establish that

$$\forall s \in \mathbb{N}, \quad q_s z_s = v_s \text{ a.e. in } \Omega_s. \tag{25}$$

Using the reflexivity of the space $W^{1,p}(\Omega)$, the boundedness of the sequence $\{z_s\}$ in $W^{1,p}(\Omega)$, and condition $(*_1)$ of Theorem 1, we find that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in W^{1,p}(\Omega)$ such that

$$z_{s_j} \to v \text{ strongly in } L^p(\Omega) \tag{26}$$

and $z_{s_j} \to v$ a.e. in Ω. The latter limit relation along with the inclusion $\{z_{s_j}\} \subset U$ implies that $v \in U$. Finally, we derive from (25) and (26) that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$. Thus, we have established that, for every sequence $v_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in U$ such that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$.

Proposition 8. *Assume that conditions* $(*_1)$ *and* $(*_2)$ *of Theorem* 1 *and condition* $(*')$ *of Theorem* 2 *are satisfied. Let* $\varphi, \psi : \Omega \to \overline{\mathbb{R}}$*, and let* $\varphi \leqslant \psi$ *a.e. in* Ω*. Let, for every* $s \in \mathbb{N}$*,* $U_s = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}$*, and let* $U = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\}$*. Assume that the set* U *is nonempty. Then the sequence* $\{U_s\}$ $\mathcal{H}$*-converges to the set* U*.*

P r o o f. Let $v \in U$. For every $s \in \mathbb{N}$, we set $w_s = q_s v$. Obviously, for every $s \in \mathbb{N}$, we have $w_s \in U_s$. It is also easy to see that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$.

Next, we fix an arbitrary sequence $v_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$. Since condition $(*_2)$ of Theorem 1 is satisfied, there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in Ω_s. Then, taking into account that condition $(*_1)$ of Theorem 1 is satisfied, we derive from Proposition 4 that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $w \in W^{1,p}(\Omega)$ such that $l_{s_j} v_{s_j} \to w$ a.e. in Ω and $\|v_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$. Let us show that $\varphi \leqslant w \leqslant \psi$ a.e. in Ω. Since, for every $s \in \mathbb{N}$, we have $v_s \in U_s$, there exists a set $E' \subset \Omega$ of measure zero such that, for every $s \in \mathbb{N}$ and for every $x \in \Omega_s \setminus E'$, we have $\varphi(x) \leqslant v_s(x) \leqslant \psi(x)$. In addition, by the properties of the operators l_s, there exists a set $E'' \subset \Omega$ of measure zero such that, for every $s \in \mathbb{N}$ and for every $x \in \Omega_s \setminus E''$, we have $(l_s v_s)(x) = v_s(x)$. It is clear that

$$s \in \mathbb{N},\, x \in \Omega_s \setminus (E' \cup E'') \Longrightarrow \varphi(x) \leqslant (l_s v_s)(x) \leqslant \psi(x). \tag{21}$$

Since $l_{s_j} v_{s_j} \to w$ a.e. in Ω, there exists a set $E''' \subset \Omega$ of measure zero such that

$$\forall x \in \Omega \setminus E''', \quad (l_{s_j} v_{s_j})(x) \to w(x). \tag{22}$$

Next, for every $k \in \mathbb{N}$, we set $E^{(k)} = \Omega \setminus \bigcup\limits_{j=k}^{\infty} \Omega_{s_j}$. In view of condition $(*')$ of Theorem 2, for every $k \in \mathbb{N}$, we have $\operatorname{meas} E^{(k)} = 0$. Therefore, setting $E = \bigcup\limits_{k=1}^{\infty} E^{(k)}$, we have $\operatorname{meas} E = 0$. Now, let $x \in \Omega \setminus (E' \cup E'' \cup E''' \cup E)$. We fix an arbitrary $\varepsilon > 0$. Since $x \in \Omega \setminus E'''$, by (22), we have $(l_{s_j} v_{s_j})(x) \to w(x)$. Consequently, there exists $k \in \mathbb{N}$ such that

$$j \in \mathbb{N},\, j \geqslant k \implies |(l_{s_j} v_{s_j})(x) - w(x)| \leqslant \varepsilon. \tag{23}$$

Since $x \in \Omega \setminus E$, there exists $j \in \mathbb{N}$, $j \geqslant k$, such that $x \in \Omega_{s_j}$. Then we derive from (21) and (23) that $\varphi(x) - \varepsilon \leqslant w(x) \leqslant \psi(x) + \varepsilon$. Hence, in view of the arbitrariness of ε, we obtain the inequality $\varphi(x) \leqslant w(x) \leqslant \psi(x)$. Therefore, $\varphi \leqslant w \leqslant \psi$ a.e. in Ω. Then $w \in U$. Thus, we have established that, for every sequence $v_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $w \in U$ such that $\|v_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$.

We now conclude that the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set U. $\square$

We note that condition $(*')$ of Theorem 2 is essential for the conclusion of Proposition 8. This is justified by the following example.

Example 4. Assume that the domain Ω and the sequence of domains Ω_s are the same as in Example 3. Then conditions $(*_1)$ and $(*_2)$ of Theorem 1 are satisfied but condition $(*')$ of Theorem 2 is not satisfied. Let $\varphi : \Omega \to \overline{\mathbb{R}}$ be the function such that, for every $x \in \Omega$, $\varphi(x) = 0$. Moreover, let $\psi : \Omega \to \overline{\mathbb{R}}$ be the function such that

$$\psi(x) = \begin{cases} 0 & \text{if } x \in B, \\ 1 & \text{if } x \in \Omega \setminus B. \end{cases}$$

in view of the inclusion $v \in V$, we have $w \in V$. Consequently, $U \subset V$. In the same way, we prove that $V \subset U$. Thus, $U = V$. □

Remark 1. In the proof of Proposition 6, concerning the considered sets in $W^{1,p}(\Omega)$, we implicitly assumed that functions equivalent to elements of these sets belong to the same sets.

Proposition 7. *Assume that the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact and the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$. Then the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $W^{1,p}(\Omega)$.*

P r o o f. Let $v \in W^{1,p}(\Omega)$. For every $s \in \mathbb{N}$, we set $w_s = q_s v$. Obviously, for every $s \in \mathbb{N}$, we have $w_s \in W^{1,p}(\Omega_s)$. It is also easy to see that $\sup_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$. Next, taking a sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\sup_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, in view of the assumptions of this proposition, we deduce from Proposition 4 that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a functon $v \in W^{1,p}(\Omega)$ such that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$. Now, by Definition 6, we conclude that the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $W^{1,p}(\Omega)$. □

We note that condition $(*')$ of Theorem 2 is essential for the conclusion of Proposition 6. This is justified by the following simple example.

Example 3. Assume that Ω is a Lipschitz domain. Then the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact. Let B be a closed ball in $\mathbb{R}^n$ such that $B \subset \Omega$, and assume that, for every $s \in \mathbb{N}$, $\Omega_s = \Omega \setminus B$. In view of the known extension results for Sobolev spaces (see, for instance, [25, Theorem 7.25]), there exists a linear continuous operator $l : W^{1,p}(\Omega \setminus B) \to W^{1,p}(\Omega)$ such that, for every function $v \in W^{1,p}(\Omega \setminus B)$, we have $lv = v$ in $\Omega \setminus B$. Setting, for every $s \in \mathbb{N}$, $l_s = l$, we find that the sequence $\{l_s\}$ has all the properties described in Definition 3. Therefore, the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$. Thus, Proposition 7 implies that the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $W^{1,p}(\Omega)$. Now, let y and r be the center and the radius of the ball B, respectively, and let $B_0 = \{x \in \mathbb{R}^n : |x - y| \leqslant r/2\}$. We define

$$U = \{v \in W^{1,p}(\Omega) : v = 0 \ \text{ a.e. in } \ B_0\}.$$

It is easy to see that, for every function $v \in U$, there exists a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $\sup_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$. Next, we fix an arbitrary sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\sup_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$. Since the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $W^{1,p}(\Omega)$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in W^{1,p}(\Omega)$ such that

$$\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0. \tag{20}$$

Let φ be a function in $C_0^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$ in Ω, $\varphi = 1$ in B_0, and $\varphi = 0$ in $\Omega \setminus B$. We have $v\varphi \in W^{1,p}(\Omega)$. Then, since $\varphi = 1$ in B_0, we have $v - v\varphi \in U$. Moreover, taking into account that $\varphi = 0$ in $\Omega \setminus B$, we derive from (20) that $\|v_{s_j} - q_{s_j}(v - v\varphi)\|_{L^p(\Omega_{s_j})} \to 0$. Now, we conclude that the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set U. Obviously, $U \neq W^{1,p}(\Omega)$. It remains to observe that $\Omega \setminus \bigcup_{s=1}^{\infty} \Omega_s = B$. Hence, $\operatorname{meas}\Big(\Omega \setminus \bigcup_{s=1}^{\infty} \Omega_s\Big) > 0$. Consequently, condition $(*')$ of Theorem 2 is not satisfied.

We now proceed to a more delicate question on the $\mathcal{H}$-convergence of sets defined by bilateral constraints.

Proposition 5. *Let condition* $(*_1)$ *of Theorem* 1 *be satisfied, and assume that there exists a sequence of linear continuous operators* $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ *such that the sequence of norms* $\|l_s\|$ *is bounded and, for every* $s \in \mathbb{N}$ *and for every* $v \in W^{1,p}(\Omega_s)$*, we have* $q_s(l_s v) = v$ *a.e. in* Ω_s*. In addition, assume that condition* $(*')$ *of Theorem* 2 *is satisfied. Let, for every* $s \in \mathbb{N}$*,* $w_s \in W^{1,p}(\Omega_s)$*, and let* $w \in W^{1,p}(\Omega)$*. Assume that the sequence of norms* $\|w_s\|_{W^{1,p}(\Omega_s)}$ *is bounded and* $\|w_s - q_s w\|_{L^p(\Omega_s)} \to 0$*. Then* $l_s w_s \to w$ *weakly in* $W^{1,p}(\Omega)$*.*

P r o o f. The properties of the operators l_s imply that the sequence $\{l_s w_s\}$ is bounded in $W^{1,p}(\Omega)$ and

$$\forall s \in \mathbb{N}, \quad q_s(l_s w_s) = w_s \text{ a.e. in } \Omega_s. \tag{17}$$

Assume that the sequence $\{l_s w_s\}$ does not converge weakly to w in $W^{1,p}(\Omega)$. Then there exist a functional $g \in (W^{1,p}(\Omega))^*$, a number $\varepsilon > 0$, and an increasing sequence $\{\bar{s}_k\} \subset \mathbb{N}$ such that

$$\forall k \in \mathbb{N}, \quad |\langle g, l_{\bar{s}_k} w_{\bar{s}_k}\rangle - \langle g, w\rangle| > \varepsilon. \tag{18}$$

Since the space $W^{1,p}(\Omega)$ is reflexive and the sequence $\{l_s w_s\}$ is bounded in $W^{1,p}(\Omega)$, there exist an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ and a function $w_0 \in W^{1,p}(\Omega)$ such that

$$l_{s_j} w_{s_j} \to w_0 \text{ weakly in } W^{1,p}(\Omega). \tag{19}$$

Hence, by condition $(*_1)$ of Theorem 1, we have $l_{s_j} w_{s_j} \to w_0$ strongly in $L^p(\Omega)$. Then, in view of (17), we have $\|w_{s_j} - q_{s_j} w_0\|_{L^p(\Omega_{s_j})} \to 0$. This and the assumption that $\|w_s - q_s w\|_{L^p(\Omega_s)} \to 0$ imply that $\|q_{s_j}(w - w_0)\|_{L^p(\Omega_{s_j})} \to 0$. Consequently, $\liminf\limits_{s\to\infty} \|q_s(w - w_0)\|_{L^p(\Omega_s)} = 0$. From this equality, condition $(*')$ of Theorem 2, and Proposition 3, we derive that $w = w_0$ a.e. in Ω. Then, in view of (19), we have $l_{s_j} w_{s_j} \to w$ weakly in $W^{1,p}(\Omega)$. However, this contradicts (18). The obtained contradiction proves that $l_s w_s \to w$ weakly in $W^{1,p}(\Omega)$. □

The following definition essentially is a particular case of Definition 5 in [6].

Definition 6. Let, for every $s \in \mathbb{N}$, U_s be a nonempty set in $W^{1,p}(\Omega_s)$, and let U be a nonempty set in $W^{1,p}(\Omega)$. We say that the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set U if the following conditions are satisfied:

(a) for every function $v \in U$, there exists a sequence $w_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$;

(b) for every sequence $v_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in U$ such that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$.

Proposition 6. *Let condition* $(*')$ *of Theorem* 2 *be satisfied. Then a sequence of nonempty sets* $U_s \subset W^{1,p}(\Omega_s)$ *may* $\mathcal{H}$*-converge to only one nonempty set* $U \subset W^{1,p}(\Omega)$*.*

P r o o f. Assume that a sequence of nonempty sets $U_s \subset W^{1,p}(\Omega_s)$ $\mathcal{H}$-converges to nonempty sets $U \subset W^{1,p}(\Omega)$ and $V \subset W^{1,p}(\Omega)$. Let $w \in U$. Since the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set U, there exists a sequence $w_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s w\|_{L^p(\Omega_s)} \to 0$. Since the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set V, for the sequence $\{w_s\}$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in V$ such that $\|w_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$. This convergence along with the convergence $\|w_s - q_s w\|_{L^p(\Omega_s)} \to 0$ implies that $\|q_{s_j}(v - w)\|_{L^p(\Omega_{s_j})} \to 0$. Then, taking into account condition $(*')$ of Theorem 2 and Proposition 3, we find that $w = v$ a.e. in Ω. Therefore, Hence, in view of the arbitrariness of ε, we conclude that $v = 0$ a.e. in Ω. Thus, condition (12) is satisfied.

Conversely, assume that condition (12) is satisfied. Let $\{m_j\}$ be an increasing sequence in $\mathbb{N}$. Setting $E_0 = \Omega \setminus \bigcup_{j=1}^{\infty} \Omega_{m_j}$, we suppose that $\operatorname{meas} E_0 > 0$. Let $\chi : \Omega \to \mathbb{R}$ be the characteristic function of the set E_0. Obviously, $\chi \in L^1(\Omega)$ and $\int_{\Omega_{m_j}} \chi\, dx = 0$ for every $j \in \mathbb{N}$. Therefore,

$$\liminf_{s\to\infty} \int\limits_{\Omega_s} \chi\, dx = 0.$$

Then, by condition (12), we have $\chi = 0$ a.e. in Ω. Hence, there exists a set $E \subset \Omega$ of measure zero such that, for every $x \in \Omega \setminus E$, we have $\chi(x) = 0$. Then, fixing $x \in E_0 \setminus E$, we obtain $\chi(x) = 0$. On the other hand, by the definition of the function χ, we have $\chi(x) = 1$. The obtained contradiction proves that $\operatorname{meas} E_0 = 0$. Thus, condition $(*')$ of Theorem 2 is satisfied. □

Proposition 3. *Let condition $(*')$ of Theorem 2 be satisfied. Then the following condition is satisfied*:

$$\textit{if } v \in W^{1,p}(\Omega) \textit{ and } \liminf_{s\to\infty} \|q_s v\|_{L^p(\Omega_s)} = 0, \textit{ then } v = 0 \textit{ a.e. in } \Omega. \tag{14}$$

P r o o f. Let $v \in W^{1,p}(\Omega)$ and $\liminf_{s\to\infty} \|q_s v\|_{L^p(\Omega_s)} = 0$. Setting $w = |v|^p$, we have

$$w \in L^1(\Omega), \qquad \liminf_{s\to\infty} \int\limits_{\Omega_s} w\, dx = 0. \tag{15}$$

Since, by assumption, condition $(*')$ of Theorem 2 is satisfied, we deduce from Proposition 2 that condition (12) is satisfied. The latter condition along with (15) implies that $w = 0$ a.e. in Ω. Hence, $v = 0$ a.e. in Ω. Thus, condition (14) is satisfied. □

Proposition 4. *Let condition $(*_1)$ of Theorem 1 be satisfied, and assume that there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in Ω_s. Let, for every $s \in \mathbb{N}$, $w_s \in W^{1,p}(\Omega_s)$. Assume that the sequence of norms $\|w_s\|_{W^{1,p}(\Omega_s)}$ is bounded. Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $w \in W^{1,p}(\Omega)$ such that $l_{s_j} w_{s_j} \to w$ weakly in $W^{1,p}(\Omega)$, $l_{s_j} w_{s_j} \to w$ a.e. in Ω, and $\|w_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$.*

P r o o f. The properties of the operators l_s along with the boundedness of the sequence of norms $\|w_s\|_{W^{1,p}(\Omega_s)}$ imply that the sequence $\{l_s w_s\}$ is bounded in $W^{1,p}(\Omega)$ and

$$\forall s \in \mathbb{N}, \quad q_s(l_s w_s) = w_s \text{ a.e. in } \Omega_s. \tag{16}$$

Since the space $W^{1,p}(\Omega)$ is reflexive and the sequence $\{l_s w_s\}$ is bounded in $W^{1,p}(\Omega)$, there exist an increasing sequence $\{\bar{s}_k\} \subset \mathbb{N}$ and a function $w \in W^{1,p}(\Omega)$ such that $l_{\bar{s}_k} w_{\bar{s}_k} \to w$ weakly in $W^{1,p}(\Omega)$. Hence, by condition $(*_1)$ of Theorem 1, we have $l_{\bar{s}_k} w_{\bar{s}_k} \to w$ strongly in $L^p(\Omega)$. Therefore, there exists an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ such that $l_{s_j} w_{s_j} \to w$ a.e. in Ω. It is clear that $l_{s_j} w_{s_j} \to w$ weakly in $W^{1,p}(\Omega)$ and $l_{s_j} w_{s_j} \to w$ strongly in $L^p(\Omega)$. The latter convergence along with (16) implies that $\|w_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$. □

in [22]. We also note that if ω is a domain of $\mathbb{R}^n$ such that $\overline{\omega} \subset \Omega$ and the origin is contained in ω, then there is no number $\delta^\omega > 0$ such that $\psi - \varphi \geqslant \delta^\omega$ a.e. in ω. We remark in this connection that it was shown in [22] that the G-convergence of a sequence of linear continuous divergence operators $A_s : \overset{\circ}{W}{}^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ to an operator $A : \overset{\circ}{W}{}^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ of the same form implies the weak convergence of solutions of variational inequalities with the operators A_s and the set of constraints $K(\psi_1, \psi_2) = \{v \in \overset{\circ}{W}{}^{1,2}(\Omega) : \psi_1 \leqslant v \leqslant \psi_2 \text{ a.e. in } \Omega\}$ to the solution of the corresponding variational inequality with the operator A and the same set of constraints. At the same time, it was assumed in [22] that $\psi_1, \psi_2 \in L^2(\Omega)$ and, for every subdomain $\omega \subset\subset \Omega$, there exist a number $\delta^\omega > 0$ and functions $\psi_1^\omega, \psi_2^\omega \in \overset{\circ}{W}{}^{1,2}(\Omega)$ such that $\psi_1 \leqslant \psi_1^\omega \leqslant \psi_2^\omega \leqslant \psi_2$ in Ω and $\psi_2^\omega - \psi_1^\omega \geqslant \delta^\omega$ in ω. Obviously, the functions φ and ψ defined at the beginning of this example do not satisfy the assumption given in [22].

We now discuss condition $(*')$ of Theorem 2. This condition is essential for the conclusion of Theorem 2. In [10], we construct an example where all the conditions of Theorem 2 are satisfied except for condition $(*')$ but the conclusion of this theorem does not hold. We call condition $(*')$ of Theorem 2 the exhaustion condition of the domain Ω by the domains Ω_s. This condition plays an important role in the study of the convergence of solutions of variational problems with irregular unilateral and bilateral constraints in variable domains. In this regard, in addition to the present paper, see [23, 24]. We used the same exhaustion condition earlier in [6] for the investigation of both a convergence of sets in variable Sobolev spaces and the coercivity of the Γ-limit of functionals defined on these spaces. Below, we show how such questions are solved for sequences of sets $U_s \subset W^{1,p}(\Omega_s)$ and the functionals $F_s + G_s$. Before we do this, let us give some useful results.

Proposition 2. *Condition $(*')$ of Theorem 2 is equivalent to the following condition:*

$$\text{if } v \in L^1(\Omega) \text{ and } \liminf_{s\to\infty} \int\limits_{\Omega_s} |v|dx = 0, \text{ then } v = 0 \text{ a.e. in } \Omega. \tag{12}$$

P r o o f. Assume that condition $(*')$ of Theorem 2 is satisfied. Let $v \in L^1(\Omega)$, and let

$$\liminf_{s\to\infty} \int\limits_{\Omega_s} |v|dx = 0.$$

Fixing an arbitrary $\varepsilon > 0$, we find that there exists an increasing sequence $\{s_j\} \subset \mathbb{N}$ such that

$$\forall j \in \mathbb{N}, \quad \int\limits_{\Omega_{s_j}} |v|dx \leqslant \frac{\varepsilon}{2^j}. \tag{13}$$

Setting $\Omega' = \bigcup\limits_{j=1}^{\infty} \Omega_{s_j}$, by condition $(*')$ of Theorem 2, we have $\operatorname{meas}(\Omega \setminus \Omega') = 0$. Then

$$\int\limits_{\Omega} |v|dx = \int\limits_{\Omega'} |v|dx \leqslant \sum_{j=1}^{\infty} \int\limits_{\Omega_{s_j}} |v|dx.$$

This and (13) imply that

$$\int\limits_{\Omega} |v|dx \leqslant \varepsilon.$$

holds for a function $\mu \in L^1(\Omega)$, $\mu \geqslant 0$ in Ω, and for every open cube Q of $\mathbb{R}^n$, a theorem on the Γ-compactness of the sequence $\{F_s\}$ can be proved similarly to the corresponding results in [19, 20]. Obviously, in this case, the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded. We also note that there are examples of sequences of nonnegative functions $\mu_s \in L^1(\Omega_s)$ for which condition (9) and condition $(*_3)$ of Theorem 1 are satisfied but there is no function $\mu_* : \Omega \to \mathbb{R}$ such that, for every $s \in \mathbb{N}$, $\mu_s \leqslant \mu_*$ a.e. in Ω_s. Such examples can be given with the use of the functions constructed in [21].

In connection with condition $(*_5)$ of Theorem 1, we give the following example.

Example 1. Let $a \in L^{p/(p-1)}(\Omega)$. Let $\beta_1 \in (0,1)$, let $\beta_2 > 0$, and let $\Phi : [0,+\infty) \to \mathbb{R}$ be a continuous function such that

$$\forall\, \eta \in [0,+\infty), \quad |\Phi(\eta)| \leqslant \beta_1 |\eta|^p + \beta_2. \tag{10}$$

For every $s \in \mathbb{N}$, we define the functional $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ by

$$G_s(v) = \int\limits_{\Omega_s} \{|v|^p + av\}dx + \Phi(\|v\|_{L^p(\Omega_s)}), \quad v \in W^{1,p}(\Omega_s).$$

In view of (10), for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, inequality (6) holds with constants c_3 and c_4 depending only on p, β_1, β_2, and $\|a\|_{L^{p/(p-1)}(\Omega)}$. We also note that if conditions $(*_1)$ and $(*_2)$ of Theorem 1 are satisfied, then, for every $s \in \mathbb{N}$, the functional G_s is weakly continuous. Next, assume that the following condition is satisfied:

$(*)$ there exists a nonnegative bounded measurable function $b : \Omega \to \mathbb{R}$ such that, for every open cube $Q \subset \Omega$, we have $\operatorname{meas}(Q \cap \Omega_s) \to \int_Q b\,dx$.

Now, let $G : W^{1,p}(\Omega) \to \mathbb{R}$ be the functional such that, for every function $v \in W^{1,p}(\Omega)$, we have

$$G(v) = \int\limits_{\Omega} b\{|v|^p + av\}dx + \Phi(\|b^{1/p}v\|_{L^p(\Omega)}). \tag{11}$$

Using condition $(*)$ and the continuity of the function Φ, we find that, for the sequence of functionals G_s, condition $(*_5)$ of Theorem 1 is satisfied.

We remark that if the domain Ω is Lipschitz and the domains Ω_s have a certain periodically perforated structure, then conditions $(*_1)$ and $(*_2)$ of Theorem 1 are satisfied along with condition $(*)$ in which the function b takes a constant positive value. Obviously, for such a function b, the functional G defined by (11) is strictly convex if the function Φ is nondecreasing and convex.

We emphasize the importance of condition $(*_6)$ of Theorem 1 for its conclusion. In [9], we gave an example where all the conditions of Theorem 1 are satisfied except for condition $(*_6)$ but the conclusion of this theorem does not hold on the whole. We note that, in this example, for an arbitrary pre-assigned positive ε, the measure of the set where the lower and upper constraints coincide does not exceed ε. Here is a simple example where condition $(*_6)$ of Theorem 1 is satisfied.

Example 2. Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, and let, for every $x \in \Omega$, we have $\varphi(x) = 0$ and $\psi(x) = |x|^2(1 - |x|^2)$. In view of these assumptions, we have $\varphi, \psi \in \overset{\circ}{W}{}^{1,p}(\Omega)$ and $\varphi \leqslant \psi$ in Ω. In addition, for every $x \in \Omega \setminus \{0\}$, $(\psi - \varphi)(x) > 0$. Thus, condition $(*_6)$ of Theorem 1 is satisfied. We observe that, in the case considered here, we have $V(\varphi,\psi) = \{v \in \overset{\circ}{W}{}^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\}$. Hence, for $p = 2$, the set $V(\varphi,\psi)$ has the same form as the set defined by bilateral constraints

$(*'')$ $\|\mu_s\|_{L^1(\Omega_s)} \to 0$;

Let, for every $s \in \mathbb{N}$, u_s *be a function in* $V_s(\psi)$ *minimizing the functional* $F_s + G_s$ *on the set* $V_s(\psi)$. *Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in V(\psi)$ *such that* u *minimizes the functional* $F + G$ *on the set* $V(\psi)$, $\|u_{s_j} - q_{s_j}u\|_{L^p(\Omega_{s_j})} \to 0$, *and* $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.

As for the proof of Theorem 2, we give the following remarks. Since, in general, the function ψ is irregular, we cannot use functions like the above functions $\tilde{u}_s$ in the proof of Theorem 1. Therefore, using operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ described in Definition 3, first, we find that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in W^{1,p}(\Omega)$ such that $l_{s_j}u_{s_j} \to u$ strongly in $L^p(\Omega)$ and almost everywhere in Ω. Then, to prove that $u \in V(\psi)$, along with the inclusions $u_s \in V_s(\psi)$, we use condition $(*')$ of Theorem 2 which effectively works in this situation. Similarly to the proof of Theorem 1, the most important step in the proof of Theorem 2 is to establish, for every function $v \in V(\psi)$, the existence of a sequence $w_s \in V_s(\psi)$ such that $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and inequality (8) holds. The construction of such a sequence involves the function v and a Γ-realizing sequence $\{v_s\}$ for v but does not involve the constraint ψ. To prove inequality (8), we essentially use condition $(*'')$ of Theorem 2 and the fact that $\mathrm{meas}(\{|v_s - q_s v| \geqslant \sigma_s q_s v\} \cap \{v > 0\}) \to 0$, where $\{\sigma_s\}$ is a sequence in $[0, 1)$ such that $\sigma_s \to 0$. For details, see the proof of Theorem 3.1 in [10].

The next result describes a situation where we have the convergence of the whole sequence of minimizers and of the whole sequence of minimum values.

Theorem 3. *Assume that conditions* $(*_1)$, $(*_2)$, $(*_4)$, *and* $(*_5)$ *of Theorem* 1 *are satisfied, and the functional* G *is strictly convex on the set* $V(\psi)$. *In addition, suppose that conditions* $(*')$ *and* $(*'')$ *of Theorem* 2 *are satisfied. Let, for every* $s \in \mathbb{N}$, u_s *be a function in* $V_s(\psi)$ *minimizing the functional* $F_s + G_s$ *on the set* $V_s(\psi)$. *Then there exists a unique function* $u \in V(\psi)$ *minimizing the functional* $F + G$ *on the set* $V(\psi)$ *and the following relations hold*: $\|u_s - q_s u\|_{L^p(\Omega_s)} \to 0$ *and* $(F_s + G_s)(u_s) \to (F + G)(u)$.

3. Comments to the conditions of Theorems 1–3

As is known (see, for instance, [14, Chapter 6]), condition $(*_1)$ of Theorem 1 is satisfied if Ω is a Lipschitz domain. In particular, bounded convex domains are Lipschitz domains. A more general requirement guaranteeing the fulfillment of condition $(*_1)$ is that Ω is an extension domain (see, for instance, [15, Chapter 1]).

Condition $(*_2)$ of Theorem 1 is satisfied, in particular, if the domains Ω_s have a certain perforated structure. In this regard, see, for instance, [16, Section 2].

As far as conditions $(*_3)$ and $(*_4)$ of Theorem 1 are concerned, we note the following. In the case where the functions μ_s take a constant value independent of s, theorems on conditions for the Γ-convergence of the integral functionals F_s with the integrands f_s satisfying condition (5) follow from the results of [17, 18], where the Γ-convergence of integral functionals defined on the spaces $W^{m,p}(\Omega_s)$ with an arbitrary $m \in \mathbb{N}$ was studied. In this case, the sequence $\{F_s\}$ Γ-converges to an integral functional defined on the space $W^{1,p}(\Omega)$, in particular, if the domains Ω_s have a periodic perforated structure and all the integrands f_s coincide with the same integrand having a certain regularity (see [17]). Obviously, in the specified case for the functions μ_s, the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded and condition $(*_3)$ of Theorem 1 is satisfied. In the more general case where $\mu_s \in L^1(\Omega_s)$ and $\mu_s \geqslant 0$ in Ω_s for every $s \in \mathbb{N}$ and, in addition, the inequality

$$\limsup_{s\to\infty} \int\limits_{Q\cap\Omega_s} \mu_s \, dx \leqslant \int\limits_{Q\cap\Omega} \mu \, dx \tag{9}$$

($*_6$) *$\psi - \varphi > 0$ a.e. in Ω.*

Let, for every $s \in \mathbb{N}$, u_s be a function in $V_s(\phi, \psi)$ minimizing the functional $F_s + G_s$ on the set $V_s(\varphi, \psi)$. Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in V(\varphi, \psi)$ such that u minimizes the functional $F + G$ on the set $V(\varphi, \psi)$, $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, and $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.

Essentially, a similar result was obtained in [12] but under stronger assumptions on the functionals F_s and G_s and under the condition $\psi - \varphi \geqslant \alpha$ a.e. in Ω, where $\alpha > 0$. In this connection, see also [13, Theorem 2.9].

Concerning the proof of Theorem 1, we note the following. First, using operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ described in Definition 3 and defining the functions $\tilde{u}_s = \min\{\max\{l_s u_s, \varphi\}, \psi\}$, we find that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in W^{1,p}(\Omega)$ such that $\tilde{u}_{s_j} \to u$ strongly in $L^p(\Omega)$ and almost everywhere in Ω. Then we obtain the inclusion $u \in V(\varphi, \psi)$, the limit relation $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, and, by virtue of conditions ($*_4$) and ($*_5$) of Theorem 1, the inequality $\liminf\limits_{s\to\infty}(F_{s_j} + G_{s_j})(u_{s_j}) \geqslant (F + G)(u)$. The next and most important step is to establish, for every function $v \in V(\varphi, \psi)$, the existence of a sequence $w_s \in V_s(\varphi, \psi)$ with the following properties: $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and

$$\limsup_{s\to\infty} F_s(w_s) \leqslant F(v). \tag{8}$$

The construction of such a sequence involves the function v and a Γ-realizing sequence $\{v_s\}$ for v, i.e., a sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $F_s(v_s) \to F(v)$, which exists in view of condition ($*_4$) of Theorem 1. Moreover, it involves the difference $\psi - \varphi$. Using the limit relation $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and condition ($*_6$) of Theorem 1, we find that, for a sequence $\{\sigma_s\} \subset (0, 1]$ converging to 0, $\operatorname{meas}\{|v_s - q_s v| \geqslant \sigma_s q_s(\psi - \varphi)\} \to 0$. This is a key moment in the proof of inequality (8). For further details leading to the required properties of the function u, see [9, Section 2].

We now proceed to the case of irregular bilateral constraints. More precisely, we consider the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative function. Thus, in contrast to the previous case, the upper constraint can be irregular and both constraints can coincide on a set of positive measure. This is due to an additional condition on the domains Ω_s and a stronger condition on the functions μ_s as compared to condition ($*_3$) of Theorem 1.

Let $\psi : \Omega \to \overline{\mathbb{R}}$ and $\psi \geqslant 0$ a.e. in Ω. We define

$$V(\psi) = \{v \in W^{1,p}(\Omega) : 0 \leqslant v \leqslant \psi \text{ a.e. in } \Omega\},$$

and let, for every $s \in \mathbb{N}$,

$$V_s(\psi) = \{v \in W^{1,p}(\Omega_s) : 0 \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}.$$

It is easy to see that the set $V(\psi)$ is nonempty, closed, and convex. Moreover, for every $s \in \mathbb{N}$, the set $V_s(\psi)$ is nonempty, closed, and convex.

Obviously, for every $s \in \mathbb{N}$, there exists a function belonging to the set $V_s(\psi)$ and minimizing the functional $F_s + G_s$ on this set.

Theorem 2. *Assume that conditions ($*_1$), ($*_2$), ($*_4$), and ($*_5$) of Theorem 1 are satisfied. In addition, suppose that the following conditions are satisfied:*

($*'$) *for every increasing sequence $\{m_j\} \subset \mathbb{N}$, we have* $\operatorname{meas}\Big(\Omega \setminus \bigcup\limits_{j=1}^{\infty} \Omega_{m_j}\Big) = 0$;

In view of the assumptions on the functions f_s and μ_s, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, the function $f_s(x, \nabla v)$ is summable on Ω_s.

Definition 5. If $s \in \mathbb{N}$, then $F_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ is the functional such that, for every function $v \in W^{1,p}(\Omega_s)$, we have
$$F_s(v) = \int\limits_{\Omega_s} f_s(x, \nabla v)dx.$$

By virtue of the conditions on the functions f_s, for every $s \in \mathbb{N}$, the functional F_s is convex and locally bounded. Therefore, for every $s \in \mathbb{N}$, the functional F_s is weakly lower semicontinuous.

Let $c_3, c_4 > 0$, and let, for every $s \in \mathbb{N}$, $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ be a weakly continuous functional. We assume that, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$,
$$G_s(v) \geqslant c_3 \|v\|^p_{L^p(\Omega_s)} - c_4. \tag{6}$$

Obviously, for every $s \in \mathbb{N}$, the functional $F_s + G_s$ is weakly lower semicontinuous. Moreover, in view of (5) and (6) and the boundedness of the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$, there exist positive constants c_5 and c_6 such that, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have
$$(F_s + G_s)(v) \geqslant c_5 \|v\|^p_{W^{1,p}(\Omega_s)} - c_6. \tag{7}$$

Thus, in view of the known results on the existence of minimizers of functionals (see, for instance, [11]), if $s \in \mathbb{N}$ and U_s is a sequentially weakly closed set in $W^{1,p}(\Omega_s)$, then there exists a minimizer of the functional $F_s + G_s$ on the set U_s.

2. Variational problems with bilateral constraints

First, we consider the case of regular bilateral constraints.

Let $\varphi, \psi \in W^{1,p}(\Omega)$, and let $\varphi \leqslant \psi$ a.e. in Ω. We define
$$V(\varphi, \psi) = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\},$$
and let, for every $s \in \mathbb{N}$,
$$V_s(\varphi, \psi) = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}.$$

It is easy to see that the set $V(\varphi, \psi)$ is nonempty, closed, and convex. Similarly, for every $s \in \mathbb{N}$, the set $V_s(\varphi, \psi)$ is nonempty, closed, and convex.

Clearly, for every $s \in \mathbb{N}$, there exists a function belonging to the set $V_s(\varphi, \psi)$ and minimizing the functional $F_s + G_s$ on this set.

Theorem 1. *Assume that the following conditions are satisfied*:

$(*_1)$ *the embedding of* $W^{1,p}(\Omega)$ *into* $L^p(\Omega)$ *is compact*;

$(*_2)$ *the sequence of spaces* $W^{1,p}(\Omega_s)$ *is strongly connected with the space* $W^{1,p}(\Omega)$;

$(*_3)$ *for every sequence of measurable sets* $H_s \subset \Omega_s$ *such that* $\operatorname{meas} H_s \to 0$, *we have*
$$\int\limits_{H_s} \mu_s \, dx \to 0;$$

$(*_4)$ *the sequence* $\{F_s\}$ Γ-*converges to a functional* $F : W^{1,p}(\Omega) \to \mathbb{R}$;

$(*_5)$ *there exists a functional* $G : W^{1,p}(\Omega) \to \mathbb{R}$ *such that, for every function* $v \in W^{1,p}(\Omega)$ *and for every sequence* $v_s \in W^{1,p}(\Omega_s)$ *with the property* $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$, *we have* $G_s(v_s) \to G(v)$;

on the Sobolev spaces $W^{1,p}(\Omega_s)$, where $\{\Omega_s\}$ is a sequence of domains contained in a bounded domain Ω of $\mathbb{R}^n$. Essentially, the mentioned idea goes back to [8]. In this connection, see also [5–7].

The main content of this paper is organized as follows. In Section 1, we state the initial assumptions and the necessary definitions. In Section 2, we present our results on the convergence of minimizers and minimum values of integral and more general functionals on sets of functions defined by bilateral constraints in variable domains. We consider the case of regular constraints, i.e., constraints lying in the corresponding Sobolev space (see [9]), and the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative function (in this connection, see [10]). In both cases, a certain connection of the spaces $W^{1,p}(\Omega_s)$ with the space $W^{1,p}(\Omega)$ and the Γ-convergence of functionals defined on the spaces $W^{1,p}(\Omega_s)$ to a functional defined on $W^{1,p}(\Omega)$ are essentially used. At the same time, some other conditions on the involved domains, integrands, and constraints are also important for our convergence results. On the whole, the conditions providing these results are discussed in Section 3, where a special attention is paid to the so-called exhaustion condition of the domain Ω by the domains Ω_s. This condition is the requirement that, for every increasing sequence $\{m_j\} \subset \mathbb{N}$, the measure of the union of all the domains Ω_{m_j} is equal to the measure of the domain Ω. We also consider the notion of $\mathcal{H}$-convergence of sequences of sets $U_s \subset W^{1,p}(\Omega_s)$ to a set $U \subset W^{1,p}(\Omega)$ and show the importance of the exhaustion condition for the $\mathcal{H}$-convergence of sets of functions defined by irregular bilateral constraints.

1. Assumptions and definitions

Let $n \in \mathbb{N}$, $n \geqslant 2$, let Ω be a bounded domain of $\mathbb{R}^n$, and let $p > 1$. Let $\{\Omega_s\}$ be a sequence of domains of $\mathbb{R}^n$ contained in Ω.

It is easy to see that if $v \in W^{1,p}(\Omega)$ and $s \in \mathbb{N}$, then $v|_{\Omega_s} \in W^{1,p}(\Omega_s)$.

Definition 2. If $s \in \mathbb{N}$, then $q_s : W^{1,p}(\Omega) \to W^{1,p}(\Omega_s)$ is the mapping such that, for every function $v \in W^{1,p}(\Omega)$, we have $q_s v = v|_{\Omega_s}$.

Definition 3. We say that the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$ if there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that:

(a) the sequence of norms $\|l_s\|$ is bounded;

(b) for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in Ω_s.

The prototype of the notion in Definition 3 is the condition of strong connectedness of n-dimensional domains introduced in [8].

Definition 4. Let, for every $s \in \mathbb{N}$, $I_s : W^{1,p}(\Omega_s) \to \mathbb{R}$, and let $I : W^{1,p}(\Omega) \to \mathbb{R}$. We say that the sequence $\{I_s\}$ Γ-converges to the functional I if the following conditions are satisfied:

(a) for every function $v \in W^{1,p}(\Omega)$, there exists a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $I_s(w_s) \to I(v)$;

(b) for every function $v \in W^{1,p}(\Omega)$ and for every sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$, we have $\liminf_{s\to\infty} I_s(v_s) \geqslant I(v)$.

Next, let $c_1, c_2 > 0$, and let, for every $s \in \mathbb{N}$, $\mu_s \in L^1(\Omega_s)$ and $\mu_s \geqslant 0$ in Ω_s. We assume that the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded.

Let, for every $s \in \mathbb{N}$, $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions: for every $\xi \in \mathbb{R}^n$, the function $f_s(\cdot, \xi)$ is measurable on Ω_s; for almost every $x \in \Omega_s$, the function $f_s(x, \cdot)$ is convex on $\mathbb{R}^n$; for almost every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$, we have

$$c_1|\xi|^p - \mu_s(x) \leqslant f_s(x, \xi) \leqslant c_2|\xi|^p + \mu_s(x). \tag{5}$$

and minimum values. A simple version of the variational property of the Γ-convergence is the following proposition.

Proposition 1. *Let, for every $s \in \mathbb{N}$, $f_s : \mathbb{R} \to \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$. Assume that the sequence $\{f_s\}$ Γ-converges to the function f. Let, for every $s \in \mathbb{N}$, x_s be a minimizer of f_s on $\mathbb{R}$. Assume that $x_s \to x$. Then x minimizes f on $\mathbb{R}$ and $f_s(x_s) \to f(x)$.*

P r o o f. Since $x_s \to x$, by condition (b) in Definition 1, we have

$$\liminf_{s\to\infty} f_s(x_s) \geqslant f(x). \tag{1}$$

Now, let $y \in \mathbb{R}$. By virtue of condition (a) in Definition 1, there exists a sequence $\{y_s\} \subset \mathbb{R}$ such that

$$f_s(y_s) \to f(y). \tag{2}$$

Since, for every $s \in \mathbb{N}$, x_s minimizes f_s on $\mathbb{R}$, we have

$$\forall s \in \mathbb{N}, \quad f_s(x_s) \leqslant f_s(y_s). \tag{3}$$

Relations (2) and (3) imply that

$$\limsup_{s\to\infty} f_s(x_s) \leqslant f(y). \tag{4}$$

From (1) and (4), we derive that x minimizes f on $\mathbb{R}$ and $f_s(x_s) \to f(x)$. We note that the latter limit relation follows from inequality (1) and from inequality (4) with $y = x$. □

Here, we have restricted ourselves only to a simplest version of the variational property of the Γ-convergence, having shown how both conditions (a) and (b) in Definition 1 work. The considered case is very simple not only due the fact that we dealt with functions defined on $\mathbb{R}$ but also because of the assumption that the minimizers of these functions are global. In the case of minimizers on sets defined by certain constraints, the situation is more complicated, and not always the "global" Γ-convergence (i.e., the convergence of the kind described in Definition 1 with a Γ-realizing sequence $\{y_s\}$ taken in the whole corresponding space) can be used for the study of the convergence of such minimizers.

There are analogues of the above definition of Γ-convergence for functionals defined on a Banach space (in particular, on a Lebesgue or Sobolev space). In this connection, see, for instance, [2, 4]. The notion of Γ-convergence of functionals with varying domain of definition (in particular, of functionals $\mathcal{I}_s : W^{m,p}(\Omega_s) \to \mathbb{R}$ with taking into account the structure of domains Ω_s) was introduced and studied, for instance, in [5–7].

Next, note that, in the study of the convergence of minimizers u_s of functionals $\mathcal{I}_s : W_s \to \mathbb{R}$, a connection of the spaces W_s with a space W plays an important role. Often, this connection is expressed as the requirement that there exists a sequence of operators $l_s : W_s \to W$ with certain properties. In particular, these properties should provide the following property: for every sequence $v_s \in W_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W_s} < +\infty$, the sequence $\{l_s v_s\}$ is bounded in W. Under appropriate and in some sense natural conditions on the functionals $\mathcal{I}_s$, for the sequence of minimizers $u_s \in W_s$ of the functionals $\mathcal{I}_s$, the inequality $\sup\limits_{s\in\mathbb{N}} \|u_s\|_{W_s} < +\infty$ holds. Therefore, if there exists a sequence $l_s : W_s \to W$ with the above mentioned property, then the sequence $\{l_s u_s\}$ is bounded. Consequently, if the space W is reflexive, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and an element $u \in W$ such that $l_{s_j} u_{s_j} \to u$ weakly in W. Actually, this is the first step in the study of the convergence of the sequence of minimizers $u_s \in W_s$ of the functionals $\mathcal{I}_s$. The described idea with the operators l_s is realized in the justification of the results stated below for functionals defined

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CONVERGENCE OF SOLUTIONS OF BILATERAL PROBLEMS IN VARIABLE DOMAINS AND RELATED QUESTIONS

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Abstract: We discuss some results on the convergence of minimizers and minimum values of integral and more general functionals on sets of functions defined by bilateral constraints in variable domains. We consider the case of regular constraints, i.e., constraints lying in the corresponding Sobolev space, and the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative function. The first case concerns a larger class of integrands and requires the positivity almost everywhere of the difference between the upper and lower constraints. In the second case, this requirement is absent. Moreover, in the latter case, the exhaustion condition of an n-dimensional domain by a sequence of n-dimensional domains plays an important role. We give a series of results involving this condition. In particular, using the exhaustion condition, we prove a certain convergence of sets of functions defined by bilateral (generally irregular) constraints in variable domains.

Key words: Integral functional, Bilateral problem, Minimizer, Minimum value, Γ-convergence of functionals, Strong connectedness of spaces, $\mathcal{H}$-convergence of sets, Exhaustion condition.

Introduction

This paper is mainly based on the talk given by the author at the International S.B. Stechkin Summer Workshop-Conference on Function Theory, Miass, Russia, August 1–10, 2017.

The problems considered in the paper are related to the following general problem. Let $\{W_s\}$ be a sequence of Banach spaces, and let, for every $s \in \mathbb{N}$, $\mathcal{I}_s : W_s \to \mathbb{R}$ and $V_s \subset W_s$, $V_s \neq \varnothing$. Let, for every $s \in \mathbb{N}$, u_s be a minimizer of $\mathcal{I}_s$ on V_s. The questions are, what are general conditions under which the sequence $\{u_s\}$ converges in a certain sense to an element and this limit element minimizes a functional $\mathcal{I}$ on a set V, and how are the functional $\mathcal{I}$ and the set V related to the sequences $\{\mathcal{I}_s\}$ and $\{V_s\}$? Problems of this kind are studied in the framework of homogenization theory. There is a special kind of convergence of functionals that helps to solve the mentioned problems. This is the Γ-convergence. There are many works devoted to the study of this convergence. The Γ-convergence of functionals with the same domain of definition was studied, for instance, in [1–3]. In the simplest case, the definition of Γ-convergence is as follows.

Definition 1. Let, for every $s \in \mathbb{N}$, $f_s : \mathbb{R} \to \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$. We say that the sequence $\{f_s\}$ Γ-converges to the function f if the following conditions are satisfied:

(a) for every $x \in \mathbb{R}$, there exists a sequence $\{y_s\} \subset \mathbb{R}$ such that $y_s \to x$ and $f_s(y_s) \to f(x)$;

(b) for every $x \in \mathbb{R}$ and every sequence $\{x_s\} \subset \mathbb{R}$ such that $x_s \to x$, we have the inequality $\liminf\limits_{s\to\infty} f_s(x_s) \geqslant f(x)$.

The Γ-convergence of ordinary real functions and functionals defined on Banach spaces has some interesting properties that distinguish it from other kinds of convergence of the corresponding mappings. Among various properties of the Γ-convergence, we only mention its variational property that describes the relation of this convergence of functionals to the convergence of their minimizers

Covering the whole rectangle $\big[0, \pi/a_{k_{i+1}-1}\big] \times [-c_{k_i}, c_{k_i}]$ and using (6), we can obtain the estimate

$$\nu(f, \pi\varepsilon)_{\left[0,\, \pi/a_{k_{i+1}-1}\right]} \leqslant \left\lceil \frac{\pi}{a_{k_{i+1}-1}\pi\varepsilon} \right\rceil \left\lceil \frac{2c_{k_i}}{\pi\varepsilon} \right\rceil \leqslant \frac{8}{a_{k_{i+1}-1}\pi\varepsilon^2} \leqslant \frac{24}{\pi}\mu\big(\pi\varepsilon\big); \tag{14}$$

here and in what follows, $\lceil x \rceil$ stands for the rounding of x upward.

It remains to cover the graph on the interval $\big[\pi/a_{k_i},\, \pi/a_{k_i-1}\big]$ where $f(x) = c_{k_i} \sin a_{k_i} x$. We can divide this interval into $N_i = 2a_{k_i}/a_{k_i-1} - 2$ intervals of monotonicity of f: $\big[\pi/a_{k_i} + \pi(n-1)/2a_{k_i}, \pi/a_{k_i} + \pi n/2a_{k_i}\big]$, $n = 1, \ldots, N_i$. Let us show that, to cover the graph of f on each of these intervals, we need at most $8/\pi\varepsilon$ squares. Using the definition of the length of a curve, we can show that the length of the graph of f on these intervals is at most $\pi/2a_{k_i} + 2c_{k_i}$. Squares with sides of length $\pi\varepsilon$ can cover the graph of a monotone function of length at least $\pi\varepsilon$. Hence,

$$\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i} + \pi(n-1)/2a_{k_i},\, \pi/a_{k_i} + \pi n/2a_{k_i}\right]} \leqslant \left\lceil \left(\frac{\pi}{2a_{k_i}} + 2c_{k_i}\right) \frac{1}{\pi\varepsilon} \right\rceil \leqslant \frac{8}{\pi\varepsilon}.$$

From (6) and the monotonicity of $\varepsilon\mu(\varepsilon)$, we obtain

$$\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i},\, \pi/a_{k_i-1}\right]} \leqslant \frac{4a_{k_i}}{\pi\varepsilon a_{k_i-1}} \leqslant \frac{12\mu\left(\dfrac{\pi}{a_{k_i}}\right)}{\pi\varepsilon a_{k_i}} = \frac{12\mu\left(\dfrac{\pi}{a_{k_i}}\right)\dfrac{\pi}{a_{k_i}}\mu(\pi\varepsilon)}{\pi^2\varepsilon\mu(\pi\varepsilon)} \leqslant \frac{12}{\pi}\mu(\pi\varepsilon). \tag{15}$$

Finally, by (12), (13), (14), and (15), we obtain the following estimate for the modulus of fractality of f:

$$\begin{aligned} \nu(f, \pi\varepsilon) \leqslant 2\nu(f, \pi\varepsilon)_{[0,\pi]} \leqslant 2\Big(&\nu(f, \pi\varepsilon)_{\left[0,\, \pi/a_{k_{i+1}-1}\right]} + \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_{i+1}-1},\, \pi/a_{k_i}\right]} \\ &+ \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i},\, \pi/a_{k_i-1}\right]} + \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i-1},\, \pi/a_{k_{i-1}}\right]} + \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_{i-1}},\, \pi\right]}\Big) \\ &\leqslant 2\left(\frac{24}{\pi}\mu\big(\pi\varepsilon\big) + \frac{\pi}{\varepsilon} + \frac{12}{\pi}\mu(\pi\varepsilon) + \mu(\pi\varepsilon)\right) = O\big(\mu(\pi\varepsilon)\big), \end{aligned}$$

i.e., $f \in F^{\mu}$.

The theorem is proved.

REFERENCES

1. **Bari N.K.** Trigonometric series. Moscow: Fizmatgiz, 1961. 936 p. [in Russian].
2. **Salem R.** Essais sur les séries trigonométriques. Actual. Sci. et Industr. Vol. 862, 1940.
3. **Waterman D.** On converges of Fourier series of functions of generalized bounded variation // Studia Mathematica, 1972. Vol. 44. P. 107–117.
4. **Gridnev M. L.** About classes of functions with a restriction on the fractality of their graphs // CEUR-WS Proceedings, 2017. Vol.1894: Proceedings of the 48th Intern. Youth School-Conf.: Modern Problems in Mathematics and its Applications, Ekaterinburg, February 5–11, 2017. P. 167–173. http://ceur-ws.org/Vol-1894/appr5.pdf [in Russian].

Suppose that $k_1, \dots, k_{i-1}$ have been already defined. Then the function $f(t)/t$ is defined, bounded, and continuous on $\left(\pi/a_{k_i-1}, \pi\right]$. Extending this function by zero to $[-\pi, \pi]$ and assuming that k_i are large enough (this is the first of two conditions on k_i), we can make the Fourier coefficient a_{k_i} of the obtained function small enough; more precisely,

$$|J_i'''| = \left| \int_{\pi/a_{k_i-1}}^{\pi} \frac{f(t)}{t} \sin a_{k_i} t \, dt \right| \leqslant \frac{1}{i}. \tag{10}$$

It remains to estimate J_i''. We have

$$\begin{aligned} J_i'' &= \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} c_{k_i} \sin a_{k_i} t \frac{\sin a_{k_i} t}{t} dt = \frac{c_{k_i}}{2} \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} \frac{1 - \cos 2a_{k_i} t}{t} dt \\ &= \frac{c_{k_i}}{2} \ln \frac{a_{k_i}}{a_{k_i-1}} - \frac{c_{k_i}}{2} \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} \frac{\cos 2a_{k_i} t}{t} dt. \end{aligned}$$

According to the second mean value theorem, taking into account that the function $1/t$ is positive and monotone, we find that

$$\left| \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} \frac{\cos 2a_{k_i} t}{t} dt \right| \leqslant \frac{a_{k_i}}{\pi} \left| \int_{\pi/a_{k_i}}^{\xi} \cos 2a_{k_i} t \, dt \right| \leqslant \frac{a_{k_i}}{\pi} \frac{2}{2a_{k_i}} = \frac{1}{\pi}.$$

Thus,

$$J_i'' = \frac{c_{k_i}}{2} \ln \frac{a_{k_i}}{a_{k_i-1}} + o(1). \tag{11}$$

Combining (8), (9), (10), and (11), and taking into account (7), we conclude that

$$J_i = \frac{c_{k_i}}{2} \ln \frac{a_{k_i}}{a_{k_i-1}} + o(1) = \frac{1}{2} \sqrt{\ln \frac{a_{k_i}}{a_{k_i-1}}} + o(1) \to +\infty.$$

Let us now estimate the modulus of fractality $\nu(f, \varepsilon)$. Denote by $\nu(f, \varepsilon)_{[a,b]}$ the minimal number of squares with sides of length ε parallel to the coordinate axes that cover the graph of the function f on $[a, b]$.

If $k_1, \dots, k_{i-1}$ have been already defined, then the function f is defined on the interval $[\pi/a_{k_{i-1}}, \pi]$ and has bounded variation; hence, by (2),

$$\nu(f, \varepsilon)_{\left[\pi/a_{k_{i-1}}, \pi\right]} = O\left(\frac{1}{\varepsilon}\right).$$

Condition (3) allows us to take k_i such that, for $\pi\varepsilon \in (0, \pi/a_{k_i}]$,

$$\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_{i-1}}, \pi\right]} \leqslant \mu(\pi\varepsilon). \tag{12}$$

This is the second condition on k_i.

Let $0 < \varepsilon \leqslant 1$. Then there exists $i \in \mathbb{N}$ such that $\varepsilon \in [1/a_{k_{i+1}}, 1/a_{k_i}]$. Let us prove the inequality $\nu(f, \pi\varepsilon) \leqslant C\mu(\pi\varepsilon)$ with some constant C. It follows from what is proved above that the required inequality holds for the covering of the graph on $[\pi/a_{k_{i-1}}, \pi]$. The inequality also holds for the intervals $[\pi/a_{k_{i+1}-1}, \pi/a_{k_i}]$ and $[\pi/a_{k_i-1}, \pi/a_{k_{i-1}}]$ where f is identically zero; hence,

$$\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_{i+1}-1}, \pi/a_{k_i}\right]} + \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i-1}, \pi/a_{k_{i-1}}\right]} \leqslant \frac{\pi}{\varepsilon}. \tag{13}$$

Consider the half-open intervals

$$I_k = \left(\frac{\pi}{a_k}, \frac{\pi}{a_{k-1}}\right], \qquad k \in \mathbb{N}.$$

Let $\{k_i\}_{i=0}^{\infty}$, $k_0 = 1$, be an increasing sequence, on which, in what follows, two additional conditions will be imposed. Let

$$c_k = \begin{cases} \sqrt{\dfrac{1}{\ln a_k/a_{k-1}}}, & k \in \{k_i\}_{i=0}^{\infty}; \\ 0, & k \notin \{k_i\}_{i=0}^{\infty}. \end{cases}$$

Finally, we define the function f on the interval $[-\pi, \pi]$:

$$f(x) = c_k \sin a_k x, \qquad x \in I_k,$$
$$f(0) = 0,$$
$$f(-x) = f(x).$$

We extend the function f to $\mathbb{R}$ periodically. The resulting function is continuous on each I_k and, since a_k/a_{k-1} is integer, is continuous and vanishes at the points $\pm\pi/a_k$. Thus, the function f is continuous on $[-\pi, \pi]$.

Since f has only a finite number of maxima and minima on $[\delta, \pi]$, $\delta > 0$, it has bounded variation on this interval (and on $[-\pi, -\delta]$ as well). Thus, its Fourier series converges at every $x \in [-\pi, \pi]\backslash\{0\}$.

Consider now the sequence of partial sums of the Fourier series of f at the point $x = 0$. As is known [1, Ch. 1, Sect. 32, formula (32.5)], for the function f, we have

$$S_k(x, f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin kt}{t}dt + o(1);$$

hence, for $x = 0$,

$$S_k(0, f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin kt}{t}dt + o(1).$$

The function f is even; therefore,

$$S_k(0, f) = \frac{2}{\pi}\int_{0}^{\pi} f(t)\frac{\sin kt}{t}dt + o(1).$$

Let us show that, after an appropriate choice of $\{k_i\}$,

$$J_i = \int_0^{\pi} f(t)\frac{\sin a_{k_i}t}{t}dt \to +\infty, \quad i \to +\infty.$$

Then $S_{a_{k_i}}(0, f) \to +\infty$ as $i \to +\infty$, i.e., the Fourier series of f diverges at $x = 0$.

To estimate J_i, we divide it into three terms:

$$J_i = \int_0^{\pi/a_{k_i}} f(t)\frac{\sin a_{k_i}t}{t}dt + \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} f(t)\frac{\sin a_{k_i}t}{t}dt + \int_{\pi/a_{k_i-1}}^{\pi} f(t)\frac{\sin a_{k_i}t}{t}dt = J_i' + J_i'' + J_i'''. \quad (8)$$

We have

$$\left|\frac{\sin a_{k_i}t}{t}\right| \leqslant a_{k_i}.$$

Hence,

$$|J_i'| \leqslant \max_{0 \leqslant t \leqslant \pi/a_{k_i}} |f(t)| a_{k_i}\frac{\pi}{a_{k_i}} = \pi c_{k_{i+1}} = o(1). \quad (9)$$

The following statements were proved in [4]:

$$BV = BV_1 = F_1 \quad [4, \text{Theorem } 1]; \tag{2}$$

$$BV_p \subset F_{2-1/p}, \quad p > 1 \quad [4, \text{Theorem } 2].$$

The latter is unimprovable; i.e., $BV_{p+\varepsilon} \not\subseteq F_{2-1/p}$ for all $\varepsilon > 0$.

In the present paper, we study the pointwise behavior of the Fourier series of continuous functions from F^μ.

Theorem 1. *Let $\mu\colon (0, +\infty) \to \mathbb{R}$ be a nonincreasing continuous function, let $\varepsilon\mu(\varepsilon)$ be a nonincreasing function, and let*

$$\lim_{\varepsilon\to+0} \varepsilon\mu(\varepsilon) = +\infty. \tag{3}$$

Then there exists a continuous function F^μ whose Fourier series does not converge everywhere.

P r o o f of Theorem 1. We will require that

$$\varepsilon^{-1} < \mu(\pi\varepsilon) \leqslant 2\varepsilon^{-\frac{3}{2}}, \quad \varepsilon \in (0, 1]. \tag{4}$$

By (3), the former inequality holds on an interval $(0, \delta)$ and, changing the function μ on the interval $(\frac{\delta}{2}, 1)$, we will obtain the same class F^μ. The latter inequality can only reduce the class F^μ. Thus, if we find a required function in the narrower class, it will belong to the wider class immediately.

To obtain a function $f \in F^\mu$ with divergent Fourier series, we modify Lebesgue's example from [1, Ch. 1, Sect. 46]. We start with defining an increasing sequence of natural numbers $\{a_k\}$ as follows. Let $a_0 = 1$. Suppose that the first k elements $a_0, a_1, \ldots, a_{k-1}$ have been already defined.

From inequalities (4), it follows that

$$\frac{a_{k-1}^2}{a_{k-1}} < 3\mu\left(\frac{\pi}{a_{k-1}}\right)$$

and, for $b \geqslant (6a_{k-1})^2$,

$$\frac{b^2}{a_{k-1}} \geqslant 3\mu\left(\frac{\pi}{b}\right).$$

Then, by continuity, there exists the smallest number a such that

$$\frac{a^2}{a_{k-1}} = 3\mu\left(\frac{\pi}{a}\right).$$

As a_k, we take the largest integer such that $a_k \leqslant a$ and the fraction a_k/a_{k-1} is integer. It is not hard to understand that a_k belongs to $[a - a_{k-1}, a]$, and, in view of the inequalities

$$\frac{a_k}{a_{k-1}} \geqslant \frac{a - a_{k-1}}{a_{k-1}} = 3\mu\left(\frac{\pi}{a}\right)\frac{1}{a} - 1 \geqslant 2, \tag{5}$$

we conclude that $a_k > a_{k-1}$.

The definition of a_k implies the inequality

$$\frac{1}{\varepsilon^2 a_{k-1}} \leqslant 3\mu(\pi\varepsilon), \quad \varepsilon \in \left[\frac{1}{a_k}, 1\right]. \tag{6}$$

The definition of a_k, inequalities (5), and condition (3) imply that

$$\frac{a_k}{a_{k-1}} \to +\infty, \quad k \to +\infty. \tag{7}$$

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DIVERGENCE OF THE FOURIER SERIES OF CONTINUOUS FUNCTIONS WITH A RESTRICTION ON THE FRACTALITY OF THEIR GRAPHS[1]

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Abstract: We consider certain classes of functions with a restriction on the fractality of their graphs. Modifying Lebesgue's example, we construct continuous functions from these classes whose Fourier series diverge at one point, i.e. the Fourier series of continuous functions from this classes do not converge everywhere.

Key words: Trigonometric Fourier series, Fractality, Divergence at one point, Continuous functions.

Let f be a 2π-periodic integrable function, and let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx), \tag{1}$$

where

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos kt\ dt,\quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin kt\ dt,$$

be the trigonometric Fourier series of the function f. Denote by $S_n(f,x)$ the nth partial sum of (1). It is known (see [1, Ch. 1, Sect. 39]) that if f has bounded variation on the period ($f\in BV$), then its Fourier series converges everywhere on $\mathbb{R}$, and if, in addition, f is continuous on $\mathbb{R}$, then the Fourier series converges to f uniformly on $\mathbb{R}$. Salem [2] (see also [1, Ch. 4, Sect. 5]) considered the classes BV_p of functions of bounded p-variation and proved that if $f\in BV_p$, then the Fourier series of f also converges everywhere on $\mathbb{R}$. (Further generalizations of these results see in [3]).

The author [4] studied relations between the classes BV_p and classes of continuous functions with a restriction on the fractality of their graphs.

Definition 1. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a bounded 2π-periodic function. By the modulus of fractality of the function f, we call the function $\nu(f,\varepsilon)$ which, for all $\varepsilon>0$, gives the minimal number of closed squares with sides of length ε parallel to the coordinate axes that cover the graph of the function f on $[-\pi,\pi]$.

Definition 2. Let $\mu\colon(0,+\infty)\to\mathbb{R}$ be a nonincreasing continuous function such that $\lim\limits_{\varepsilon\to 0}\mu(\varepsilon)=+\infty$. We define the functional class

$$F^{\mu}:=\{f\in C_{2\pi}:\nu(f,\varepsilon)=O(\mu(\varepsilon))\}.$$

In the case $\mu(\varepsilon)=1/\varepsilon^{\alpha}$, where $1\leqslant\alpha\leqslant 2$, we will write F_{α} instead of $F^{1/\varepsilon^{\alpha}}$.

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which gives the estimate $\|T\| \le N$. Now we show that indeed $\|T\| = N$. Consider the family of functions $y_K = e^{-Kt}$, $K > 0$. Let f_K be the corresponding right-hand side of equation (3.3). Take an arbitrary $0 < \alpha < 1$. We have

$$\alpha N^2 \|f_K\|^2 - \|Tf_K\|^2 = \alpha N^2 \int_0^\infty (-y_K'(t) + N y_K(t))^2 dt - N^2 \int_0^\infty (y_K'(t))^2 dt$$
$$= \frac{N^2}{2K}(\alpha(K+N)^2 - K^2).$$

This expression is negative for all $0 < \alpha < \dfrac{K^2}{(N+K)^2}$ which yields $\|Tf_K\|^2 > \alpha N^2 \|f_K\|^2$. Letting K go to infinity (with fixed N) we let α approach 1, and thus obtain $\|T\| \ge N$. Consequently, $\|T\| = N$.

Note that inequality (3.5) is a strict inequality if $y \ne 0$ and, consequently, $f \ne 0$. In other words, the norm of the operator T is not attained.

It can be shown similarly that the norm of the operator $V = -\dfrac{1}{2N}(B_N + I)$ is equal to $1/N$. Since the domain $\mathcal{D}(D^2)$ of the operator D^2 is dense in $L_2(0,\infty)$, it follows that the deviation of the operator T from the differentiation operator D on the class $Q^{(2)}$ is equal to $1/N$.

Thus, the approximating operator (3.2) gives the estimate $E_N(D; Q^{(2)}) \le \dfrac{1}{N}$ in the general case (1.5) as well as in the concrete case (3.1).

REFERENCES

1. **Arestov V.V.** Approximation of unbounded operators by bounded operators and related extremal problems // Russian Math. Surveys, 1996. Vol. 51, no. 6. P. 1093–1126. DOI: 10.4213/rm1019
2. **Arestov V.V., Gabushin V.N.** Best approximation of unbounded operators by bounded operators // Russian Math. (Iz. VUZ), 1995. Vol. 39, no. 11. P. 38–63.
3. **Arestov V.V., Filatova M.A.** Best approximation of the differentiation operator in the space L_2 on the semiaxis // J. Approx. Theory, 2014. Vol. 187. P. 65–81. DOI: 10.1016/j.jat.2014.08.001
4. **Babenko V.F., Korneichuk N.P., Kofanov V.A., Pichugov S.A.** Inequalities for derivatives and their applications. Kiev: Naukova Dumka, 2003. [in Russian]
5. **Berdysheva E.E.** On the best approximation of the differentiation operator in $L_2(0,\infty)$ // East J. Approx. 1996. Vol. 2, no. 3. P. 281–287.
6. **Grubb G.** Distributions and Operators. NY: Springer Science and Business Media, 2008. 464 p. DOI: 10.1007/978-0-387-84895-2
7. **Hardy G.H., Littlewood J.E., Pólya G.** Inequalities. Cambridge University Press, 1934. 314 p.
8. **Hille E., Phillips R.S.** Functional Analysis and Semigroups. American Mathematical Society, 1957. 808 p.
9. **Kato T.** On an inequality of Hardy, Littlewood and Pólya // Adv. Math., 1971. Vol. 7. P. 217–218. DOI: 10.1016/S0001-8708(71)80002-6
10. **Kato T.** Perturbation Theory for Linear Operators. Verlag–Berlin–Heidelberg: Springer, 1995. 623 p. DOI: 10.1007/978-3-642-66282-9
11. **Stechkin S.B.** Best approximation of linear operators // Math. Notes. 1967. Vol. 1. P. 91–99. DOI: 10.1007/BF01268056

$Q^{(2)}$ is the class of functions $f \in L_2(0,\infty)$ such that f' is locally absolutely continuous on $[0,\infty)$, $f'' \in L_2(0,\infty)$, and $\|f''\| \le 1$. Problem (1.5) takes in this case the form

$$E_N(D; Q^{(2)}) = \inf_{T \in \mathscr{B}(N)} \sup_{f \in Q^{(2)}} \|f' - Tf\|. \tag{3.1}$$

It took about 20 years of research to solve the problem completely. Stechkin's inequality (1.1) and inequality (1.3) of Hardy, Littlewood and Pólya provide the lower bound

$$E_N(D; Q^{(2)}) \ge \frac{1}{2N}.$$

One of the first upper bounds for (3.1)

$$E_N(D; Q^{(2)}) \le \frac{1}{\sqrt{3}N}$$

was obtained by using a concrete approximating operator by the first named author in 1996 [5]. Problem (3.1) was fully solved only in 2014 by Arestov and the second named author [3] . Namely, they showed that

$$E_N(D; Q^{(2)}) = \frac{1}{2N}.$$

In this section, we discuss what the statement of Theorem 1 means in the concrete case (3.1) of problem (1.5). The approximating operator T used in Theorem 1 is

$$T = \frac{N}{2}(B_N - I) = NA(NI - A)^{-1}. \tag{3.2}$$

Below we will describe this operator in the special case. We consider and calculate its norm $\|T\|$ and its deviation $U(T)$ from the operator $A = D$ on the class $Q^{(2)}$.

It is not difficult to see that the operator T in the concrete case can be represented as follows. Let W be the class of functions $y \in L_2(0,\infty)$ such that y is locally absolutely continuous on $[0,\infty)$ and $y' \in L_2(0,\infty)$. For $f \in L_2(0,\infty)$, we consider the differential equation

$$-y' + Ny = f, \quad y \in W. \tag{3.3}$$

For each function $f \in L_2(0,\infty)$, equation (3.3) has a unique solution which is a real-valued function from $L_2(0,\infty)$. The operator T is defined as

$$Tf = Ny', \tag{3.4}$$

where y is the solution of the differential equation (3.3).

Integrating by parts and taking into account that $\lim\limits_{t\to\infty} y(t) = 0$, we obtain (see [3] for details) that

$$\|f\|^2 = \int_0^\infty (-y'(t) + Ny(t))^2 dt = \int_0^\infty (y'(t))^2 dt + N^2 \int_0^\infty (y(t))^2 dt + Ny^2(0).$$

It follows from (3.4) that $\|Tf\|^2 = N^2 \int\limits_0^\infty (y'(t))^2 dt$. Thus, we immediately obtain

$$\|Tf\|^2 \le N^2 \|f\|^2, \tag{3.5}$$

$$\|cx - Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 - 2c\Re(Ax, x).$$

It follows immediately that

$$\|(cI + A)x\| \leq \|(cI - A)x\|. \tag{2.1}$$

Now take $y \in \mathcal{D}((cI - A)^{-1})$. Applying (2.1) to $x = (cI - A)^{-1}y \in \mathcal{D}(A)$, we obtain

$$\|(cI + A)(cI - A)^{-1}y\| \leq \|y\|,$$

and thus $\|B_c\| \leq 1$. □

Now we are ready to prove Theorem 1.

P r o o f. We will construct a concrete approximating operator T in problem (1.5) and estimate its norm and its deviation (1.6) from the operator A on the class Q_2.

Note that all the operators we consider commute on the set $\mathcal{D}(A^2)$.

The restriction of the operator A to the set $\mathcal{D}(A^2)$ (which we will denote by the same symbol) can be represented as

$$A = \frac{N}{2}(B_N - I) - \frac{1}{2N}(B_N + I)A^2.$$

Put $T : H \to H$,

$$T = \frac{N}{2}(B_N - I).$$

Then, for the restriction of the operator $A - T$ to $\mathcal{D}(A^2)$, we have

$$A - T = -\frac{1}{2N}(B_N + I)A^2.$$

We estimate the norm of the operator T as follows:

$$\|T\| = \frac{N}{2}\|B_N - I\| \leq \frac{N}{2}(\|B_N\| + \|I\|) = N. \tag{2.2}$$

For the deviation $U(T)$ of the operator T from the operator A, we obtain that

$$U(T) = \sup_{x \in Q_2} \|(A - T)x\| \leq \sup_{x \in Q_2} \frac{1}{2N}\|B_N + I\| \cdot \|A^2x\| \leq \frac{1}{N}. \tag{2.3}$$

It follows immediately from (2.2) and (2.3) that

$$E_N(A; Q_2) \leq U(T) \leq \frac{1}{N}.$$

□

3. Approximation of the differentiation operator in the space $L_2(0, \infty)$

An important concrete case of problem (1.5) is the problem of the best approximation of the differentiation operator $Df = f'$ by bounded linear operators in the Hilbert space $L_2(0, \infty)$ of real-valued functions whose squares are integrable on $(0, \infty)$ on the class $Q^{(2)}$ defined as follows:

2. The main result

The main result of the paper is the following statement.

Theorem 1. *The best approximation* (1.5) *of the infinitesimal generator A of a strongly continuous contraction semigroup in a Hilbert space on the class Q_2 defined in* (1.4) *satisfies the inequality*

$$E_N(A; Q_2) \leq \frac{1}{N}.$$

It is known that the infinitesimal generator A of a strongly continuous contraction semigroup in a Banach space possesses the following properties:

1) The domain $\mathcal{D}(A)$ of the operator A is dense (see, e.g., [6, Lemma 14.5, p. 411]).

2) The resolvent set $\rho(A)$ of the operator A contains the right half-plane $\{\lambda \in \mathbb{C} | \, \Re\lambda > 0\}$. Moreover, $\|(A - \lambda I)^{-1}\| \leq (\Re\lambda)^{-1}$ for all $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ (e.g., [6, Theorem 14.7, p. 412]).

Furthermore, if A is the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space, we have additionally:

3) The operator A is upper semibounded, with the upper bound 0, i.e.,

$$\Re(Ax, x) \leq 0$$

for $x \in \mathcal{D}(A)$ [6, Lemma 14.9, p. 416].

The following lemma is not new. However, we will formulate and prove it for the sake of completeness.

Lemma 1. *Let A be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space H and $c > 0$. Then the operator*

$$B_c = (cI + A)(cI - A)^{-1}$$

is densely defined and bounded (*and thus can be extended to the whole space H by continuity*). *Moreover,*

$$\|B_c\| \leq 1.$$

Remark. The operator B_c is the Cayley transform of the operator A in the terminology of Kato [9], see also [10, p. 545].

P r o o f. Since $c > 0$, the operator $(cI - A)^{-1}$ is defined everywhere on H and bounded. Since A is the infinitesimal generator of a strongly continuous contraction semigroup, the operator $-A$ is m-accretive (see [10, Chapter IX, §1.4 as well as Problem 1.18, both p. 485]). Therefore, the domain $\mathcal{D}(A)$ of the operator A is equal to the range $\mathcal{R}((cI - A)^{-1})$ of the operator $(cI - A)^{-1}$ which is dense in H (see [10, Chapter V, §3.10, p. 279]). Thus, B_c is densely defined.

Now we estimate the norm of B_c. For $x \in \mathcal{D}(A)$ we have

$$\|cx + Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 + 2c\Re(Ax, x),$$

linear bounded operators from X to Y with the norm $\|T\|_{X\to Y} \le N$. The best approximation of the operator A by linear bounded operators $T \in \mathscr{B}(N)$ on the class Q is

$$E_N(A;Q) = \inf\{U(A,T,Q) : T \in \mathscr{B}(N)\},$$

where

$$U(A,T,Q) = \sup\{\|Ax - Tx\|_Y : x \in Q\}$$

is the deviation of the operator T from the operator A on the class Q.

One of the most important cases of the problem formulated above is when the class Q is defined in the following way. Let Z be a Banach space and B be a linear operator from X to Z such that $\mathcal{D}(B) \subseteq \mathcal{D}(A)$. The class Q is then defined as $Q = \{x \in X \ : \|Bx\|_Z \le 1\}$.

Stechkin [11] suggested an estimate from below for the best approximation $E_N(A;Q)$ in terms of the modulus of continuity of the operator A on the class Q defined by

$$\Phi(\delta) = \sup\{\|Ax\|_Y : x \in Q,\ \|x\|_X \le \delta\}, \quad \delta > 0.$$

Namely, Stechkin showed that

$$E_N(A;Q) \ge \sup\{\Phi(\delta) - N\delta :\ \delta > 0\}. \tag{1.1}$$

In particular, when $B = A^n$, the problem $E_N(A^k;Q)$ turned out to be closely connected to the exact constants in the Kolmogorov-type inequalities of the form

$$\|A^k x\| \le C\|x\|^{\frac{n-k}{n}}\|A^n x\|^{\frac{k}{n}}, \quad x \in \mathcal{D}(A^n), \tag{1.2}$$

with $n,k \in \mathbb{N}$, $0 < k < n$, and a certain constant C that depends on n and k.

If A is the differentiation operator, inequalities (1.2) are inequalities between the norms of the derivatives of a function. Such inequalities have been studied by a large number of authors (see [1], [2], [4] and the bibliography therein). Here we only mention that Hardy, Littlewood and Pólya [7, Chapter VII, §7.8] obtained the exact inequality

$$\|f'\|^2 \le 2\|f\|\|f''\| \tag{1.3}$$

in the space $L_2(0,\infty)$ on the class of functions $f \in L_2(0,\infty)$ such that f' is locally absolutely continuous on $(0,\infty)$, and $f'' \in L_2(0,\infty)$.

In 1971, Kato [9] proved the following result which can be considered as a generalization of (1.3). Let A be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space H. Then

$$\|Ax\|^2 \le 2\|x\|\|A^2x\|, \quad x \in \mathcal{D}(A^2).$$

In this paper, we study Stechkin's problem of the best approximation of the infinitesimal generator A of a strongly continuous contraction semigroup by bounded linear operators on the class

$$Q_2 = \{x \in \mathcal{D}(A^2) :\ \|A^2x\| \le 1\} \tag{1.4}$$

in a Hilbert space. Namely, we estimate

$$E_N(A;Q_2) = \inf\{U(T) :\ T \in \mathscr{B}(N)\}, \tag{1.5}$$

where

$$U(T) = U(A,T,Q_2) = \sup\{\|Ax - Tx\| :\ x \in Q_2\}. \tag{1.6}$$

ON THE BEST APPROXIMATION OF THE INFINITESIMAL GENERATOR OF A CONTRACTION SEMIGROUP IN A HILBERT SPACE[1]

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Abstract: Let A be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space H. We give an upper estimate for the best approximation of the operator A by bounded linear operators with a prescribed norm in the space H on the class $Q_2 = \{x \in \mathcal{D}(A^2) : \|A^2x\| \le 1\}$, where $\mathcal{D}(A^2)$ denotes the domain of A^2.

Key words: Contraction semigroup, Infinitesimal generator, Stechkin's problem.

1. Introduction

Let H be a Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, and let A be the infinitesimal generator of a strongly continuous contraction semigroup in H. For the definition and properties of the infinitesimal generator of a semigroup in a Banach space see, e.g., [6, §14.2]. Note that a strongly continuous contraction semigroup is also called a contraction semigroup of the class C_0 ([8, 9]). For an operator F on the space H, $\mathcal{D}(F)$ denotes the domain of F. We denote by I the identity operator.

In this paper, we study the so-called Stechkin's problem of the best approximation of the operator A by bounded linear operators with a prescribed norm on the class of elements $x \in \mathcal{D}(A^2)$ such that $\|A^2x\| \le 1$. We give an upper estimate for the best approximation of the operator A.

The problem we consider is a special case of the general problem of the best approximation of an unbounded operator by linear bounded ones on a certain class of elements in a Banach space. This problem first appeared in Stechkin's work in 1965–1967 [11]. The problem was studied by a number of authors (see surveys [1], [2], monograph [4], paper [3], and the bibliography therein).

Stechkin formulated this problem in a general setting as follows. Let X, Y be two Banach spaces, let A be a linear operator (in general, unbounded) from X to Y, and let $Q \subseteq \mathcal{D}(A)$ be a certain class of elements from the domain $\mathcal{D}(A)$ of the operator A. We denote by $\mathscr{B}(N)$ the set of

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3. **Pyt'ev Yu.P.** Mathematical Analysis Methods of the Experiment. Moscow: Vyssh. School, 1989. 352 p. [in Russian].
4. **Ustinov V.V., Tsurin V.A., Romashev L.N., Ovchinnikov V.V.** Mössbauer spectroscopy of interlayer boundaries in magneto-noncollinear $[^{57}Fe/Cr]_{12}/MgO(100)$ superlattices // Technical Physics Letters. 1999. Vol. 25, no 6. P. 459–461.

we can to conclude that

$$\begin{aligned}
&\frac{(b+a)^{2\nu+1}}{\pi a(2\nu)!}\max_t \frac{\partial^{2\nu}}{(\partial b)^{2\nu}}(\varphi * f)(t)\\
&= \frac{(b+a)^{2\nu+1}}{2\pi a}\max_t \frac{\partial^{2\nu}}{(\partial b)^{2\nu}} \int_{-\infty}^{\infty} f(t-x)\frac{b}{b^2+x^2}\,dx\\
&= \frac{(b+a)^{2\nu+1}}{2\pi a}\max_t \int_{-\infty}^{\infty} f(t-x)\Big[\frac{1}{(b-ix)^{2\nu+1}}+\frac{1}{(b+ix)^{2\nu+1}}\Big]\,dx\\
&= \frac{(b+a)^{2\nu+1}}{\pi a(2\nu)!}\max_t \int_{-\infty}^{\infty} f(t-x)\Big[\frac{(b+ix)^{2\nu+1}+(b-ix)^{2\nu+1}}{(b^2+x^2)^{2\nu+1}}\Big]\,dx\\
&= \frac{(b+a)^{2\nu+1}}{\pi a}\max_t \int_{-\infty}^{\infty} f(t-x)\Big[\frac{1}{(b^2+x^2)^{2\nu+1}}\Big]\sum_{s=0}^{[\frac{2\nu+1}{2}]}(-1)^s C_{k+1}^{2s}x^{2s}b^{k+1-2s}\,dx.
\end{aligned} \tag{11}$$

Let us now describe a different way for computing b_1 and A_1 which is somewhat simpler, but involves a larger number of calculations of maxima. Let

$$H_r(t) = H_r(t,b) = \frac{\partial^r}{\partial b^r}\int_{-\infty}^{\infty} f(t-x)\frac{b\,dx}{b^2+x^2}.$$

If $r = 2k-1$ and the conditions of the assertion are satisfied, then, similar to (6), the following relations can be derived for H_{2k-1} as $k\to\infty$:

$$\frac{(b+a)^{2k}}{\pi a(2k-1)!}\max_t\Big[(-1)\frac{\partial^{2k-1}}{(\partial b)^{2k-1}}[\varphi * f](t)\Big] = \frac{(b+a)^{2k}}{a}\cdot\frac{A_1 b_1}{(b+b_1)^{2k}}(1+o(1)).$$

Combining this with (6), one can find approximate values for b_1 and A_1 (in this order) from the formulas

$$b_1 = 2k\frac{\max_t |H_{2k-1}(t)|}{\max_t H_{2k}(t)} - b + o(1),$$

$$A_1 = \frac{(b+b_1)^{2k+1}\max_t H_{2k}(t)}{\pi b_1(2k)!} + o(1).$$

By choosing large k, all small values denoted as $o(1)$ can be neglected.

3. Conclusion

In this paper, the new mathematical method is described for analysis of experimental data obtained for Mössbauer spectroscopy. This method allows to find the spectral decomposition of the integral transmission as a finite sum of Lorentzians with the accuracy of calculation of their number and determinations of their parameters according to a given accuracy of experimental data.

REFERENCES

1. **Koloskov V.M., Derjaguin A.I., Vil'danova N.F., Gapontsev V.L.** Concentration and structural transformations in austenitic iron-based chromium-nickel alloys under severe plastic deformation // Physical Mesomechanics. 2006. Vol. 9, no 5. P. 97–106.
2. **Rusakov V.S.** Mossbauer spectroscopy of heterogeneous systems. Almaty, 2000. 431 p. [in Russian].

holds for all sufficiently large ν, where

$$\theta = \theta_\nu, \qquad |\theta| < 1,$$

$$a = (a_1^{(\nu)} + a_2^{(\nu)})/2 = b_1 + \delta, \quad 0 \le \delta < c/2^\nu, \quad c = a_2^{(1)} - a_1^{(1)}.$$

Taking the logarithm of both sides of equality (9) implies that

$$\begin{aligned} &\ln\Big[\frac{(b+a)^{2\nu+1}}{\pi a(2\nu)!}\max_t \frac{\partial^{2\nu}}{\partial b^{2\nu}}[\varphi * f]\Big] \\ &= \ln\frac{b_1}{a} + (2\nu+1)\ln\frac{b+a}{b+b_1} + \ln\Big[1+\theta\sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2\nu+1}\Big] + \ln A_1. \end{aligned} \tag{10}$$

Hence,

$$\Big|\ln\frac{b_1}{a}\Big| = \Big|\ln\frac{b_1 + O(\frac{1}{2^\nu})}{b_1}\Big| = \Big|\ln\Big(1+O\Big(\frac{1}{2^\nu}\Big)\Big)\Big| = O\Big(\frac{1}{2^\nu}\Big),$$

$$\Big|(2\nu+1)\ln\frac{b+a}{b+b_1}\Big| = (2\nu+1)\Big|\ln\Big(1+O\Big(\frac{1}{2^\nu}\Big)\Big)\Big| = O\Big(\frac{\nu}{2^\nu}\Big),$$

$$\Big|\ln\Big[1+\theta_\nu\sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2\nu+1}\Big]\Big| \le l|\theta_\nu|\sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2\nu+1} \le C_n\Big(\frac{b+b_1}{b+b_2}\Big)^{2\nu+1} = C_n q^{2\nu+1}.$$

Therefore, these values are small for large ν, and then the approximate equality

$$A_1 \approx A_1^{(\nu)} = \frac{(b+a_\nu)^{2\nu+1}}{\pi a_\nu (2\nu)!}\max_t\Big(\frac{\partial^{2\nu}}{(\partial b)^{2\nu}}[\varphi * f](t)\Big)$$

holds. Once the parameters x_1, b_1, and A_1 are found, we can introduce the function

$$f_1(x) = f(x) - \varphi_1(x) = f(x) - \frac{A_1 b_1^2}{b_1^2 + (x-x_1)^2},$$

repeat the same operations for this function as we did for $f(x)$, and find x_2, b_2, and A_2. Then these iterations must be continued until the number A_{n+1} becomes small enough.

Let us remark that, although the algorithm uses the differentiation of the convolution of the experimental function $f(x)$ with the function $\varphi(x,b)$, first, one can differentiate the integrand and then compute the convolution $\Big[\Big(\partial^r\varphi/\partial b^r\Big) * f\Big]$. This order is preferable, because the numerical integration is a more regular operation than the incorrect numerical differentiation, and the derivatives of $\varphi(x,b)$ are expressed analytically.

Let us now write explicitly the positive (see (9)) argument of the logarithm in the left-hand side of (10) whose maximum is to be found. Starting, as above, from the formula

$$\frac{b}{b^2+x^2} = \frac{1}{2}\Big(\frac{1}{b-ix} + \frac{1}{b+ix}\Big),$$

2. Numerical algorithm

Suppose that the function $f(x)$ has a unique local maximum, i.e., there is only one point x_1 such that

$$f(x_1) = \max_x f(x).$$

We set

$$A_1 = f(x_1)$$

and find the parameter b_1 and points

$$\widetilde{x}_1 = x_1 - b_1,$$

and

$$\widetilde{x}_2 = x_1 + b_1$$

from the condition

$$\max_x \left| f(x) - \frac{A_1 b_1^2}{b_1^2 + (x - x_1)^2} \right| = \inf_c \max_x \left| f(x) - \frac{A_1 c^2}{c^2 + (x - x_1)^2} \right|.$$

Alternatively, they can be found by keeping the value of the half-width of $f(x)$ and using the following simple formulas:

$$f(\widetilde{x}_l) = \frac{1}{2} A_1 \quad (l = 1, 2),$$

$$x_1 = \frac{1}{2}(\widetilde{x}_1 + \widetilde{x}_2),$$

$$b_1 = \frac{1}{2}(\widetilde{x}_2 - \widetilde{x}_1) \ \text{ for } \ \widetilde{x}_2 > \widetilde{x}_1.$$

If the difference $f(x) - \varphi_1(x)$ is small enough, i.e., is comparable with the accuracy of the evaluation of the function $f(x)$, the algorithm terminates by setting $f(x) = \varphi_1(x)$.

Otherwise, or, if there are several points of local maxima of $f(x)$, a different process is applied. This process is based on the asymptotic behavior of (6) and is described below.

For arbitrary $a > 0$, and $b > 0$, and for sufficiently large k, it is needed to find a point of maximum of the left-hand side of (6) and the maximum value. Further, let us consider this point as an approximate value for x_1. After that values $a_1^{(1)}$, and $a_2^{(1)}$ and a positive integer $k^{(1)}$ are found such that the left-hand side of (6) is greater than $A = \max_x f(x)$ for $k = k^{(1)}$ and $a = a_2^{(1)}$ and is smaller than ε for $k = k^{(1)}$ and $a = a_1^{(1)}$, where ε is an admissible error for computing A_1 from the right-hand side of (6). Next, the segment $[a_1^{(1)}, a_2^{(1)}]$ is divided into two equal parts. One of these parts, which satisfies assumptions analogous to those for $[a_1^{(1)}, a_2^{(1)}]$ for some $k^{(2)}$ ($\geq k^{(1)}$), is taken as the next segment $[a_1^{(2)}, a_2^{(2)}]$. After several iterations, we eventually obtain a number $k^{(\nu)}$ and a segment $[a_1^{(\nu)}, a_2^{(\nu)}]$, whose length is less than a given error $\delta > 0$ $\big(a_2^{(\nu)} - a_1^{(\nu)} = (a_2^{(1)} - a_1^{(1)})/2^\nu < \delta\big)$. Since $b_1 \in [a_1^{(l)}, a_2^{(l)}]$ $(l = 1, 2, \ldots, \nu)$, we can set

$$b_1 \approx a^{(\nu)} = \frac{a_1^{(\nu)} + a_2^{(\nu)}}{2}$$

with error at most δ.

As follows from assertions (6)–(8), the equality

$$\frac{(b+a)^{2\nu+1}}{\pi a (2\nu)!} \max_t \frac{\partial^{2\nu}}{\partial b^{2\nu}} [\varphi * f] = \frac{A_1 b_1 (b+a)^{2\nu+1}}{a (b+b_1)^{2\nu+1}} \Big[1 + \theta \sum_{s=2}^{n} \frac{A_s b_s}{A_1 b_1} \Big(\frac{b+b_1}{b+b_s} \Big)^{2\nu+1} \Big] \tag{9}$$

$$= \Big|\frac{(-1)^r (r)!}{2} \cdot \frac{\exp(i\varphi(r+1)) + \exp(-i\varphi(r+1))}{\rho^{r+1}\exp(i\varphi(r+1) - i\varphi(r+1))}\Big|$$

$$= \Big|\frac{(-1)^r (r)!}{2} \cdot \frac{\cos((r+1)\varphi) + i\sin((r+1)\varphi) + \cos((r+1)\varphi) - i\sin((r+1)\varphi)}{\rho^{r+1}}\Big|$$

$$= \Big|\frac{(-1)^r (r)!\cos((r+1)\varphi)}{\rho^{r+1}}\Big| = \Big|\frac{(-1)^r (r)!\cos((r+1)\varphi)}{(p^2+\xi^2)^{(r+1)/2}}\Big| \le \frac{r!}{p^{r+1}},$$

where $p > 0$, $\rho = (p^2+\xi^2)^{1/2}$, and $\varphi = \arg(p + i\xi)$ ($\varphi = 0$ for $\xi = 0$), and the equality is attained only at the point $\xi = 0$.

Furthermore, we assume that

$$0 < b_1 < b_2 < \cdots < b_n, \quad a > 0, \quad b > 0 \tag{5}$$

and consider only derivatives of even order when the absolute value sign in (4) can be omitted, since

$$\Phi_s(t) = \Phi_s(t, b) = \frac{A_s b_s \pi p}{p^2 + \xi^2},$$

where $p = b + b_s > 0$, and $\xi = t - x_s$.

Assertion. *Under assumption* (5), *the convolution of the functions* $f(x)$ *in* (1) *and* $\varphi(x) = \varphi(x, b)$ *in* (2) *has the following asymptotic behavior as a positive integer* k *tends to infinity*:

$$\begin{aligned}&\frac{(b+a)^{2k+1}}{\pi a (2k)!}\max_t \Big(\frac{\partial}{\partial b}\Big)^{2k}[\varphi * f] \\ = \frac{(b+a)^{2k+1}}{a} \cdot &\frac{A_1 b_1}{(b+b_1)^{2k+1}}[1 + o(1)] \to \begin{cases} \infty, & \textit{for } \ a > b_1, \\ A_1, & \textit{for } \ a = b_1, \\ 0, & \textit{for } \ 0 < a < b_1. \end{cases}\end{aligned} \tag{6}$$

Indeed, from (1), (3), and (4), it follows that

$$\frac{(b+a)^{2k+1}}{\pi a (2k)!}\max_t \Big(\frac{\partial}{\partial b}\Big)^{2k}[\varphi * f](t) = \max_t \sum_{s=1}^{n} \frac{(b+a)^{2k+1}}{\pi a (2k)!}\Big(\frac{\partial}{\partial b}\Big)^{2k}\Phi_s(t, b). \tag{7}$$

Therefore,

$$\begin{aligned}&\frac{b_1 A_1}{a}\Big(\frac{b+a}{b+b_1}\Big)^{2k+1}\Big[1 - \sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2k+1}\Big] \\ &\le \frac{(b+a)^{2k+1}}{\pi a (2k)!}\max_t \Big(\frac{\partial}{\partial b}\Big)^{2k}[\varphi * f](t) \\ &\le \frac{(b+a)^{2k+1}}{a}\Big[\frac{A b_1}{(b+b_1)^{2k}} + \sum_{s=2}^{n}\frac{A_s b_s}{(b+b_1)^{2k+1}}\Big] \\ &= \frac{b_1 A_1}{a}\Big(\frac{b+a}{b+b_1}\Big)^{2k+1}\Big[1 + \sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2k+1}\Big].\end{aligned} \tag{8}$$

Now, restrictions (5) and inequalities (8) imply (6).

Remarks. By the assertion, for large k, the left-hand side of (6) is close to the maximum of the first term ($s = 1$) of the sum in (7) attained by (4) for $s = 1$ at the point $t = x_1$. Hence, if k is sufficiently large, we can approximate the value x_1 by a maximum point of the left-hand side of (7) which actually coincides with the left-hand side of (6).

In practice, before the experimental data processing, one elaborates a model of the Mössbauer spectrum. Such a model is usually based on some additional information on the concentration, the structural state, and the charge state of Mössbauer atoms in the material. Then, taking into account physical restrictions, one forms the shape of the lines which are close (in the sense of the minimum of χ^2-criterion) to the normalized experimental Mössbauer spectrum. This is done by choosing the remaining free parameters of the model. However, this approach does not guarantee a proper, physically approved model. A reasonable way to construct a structural NGR spectrum decomposition should be based on its model-free analysis. This can be done directly by solving problem (1) and by further improving the model based on the results of other methods [1].

Some model-free methods of the NGR spectra analysis have been implemented in algorithms for the density distribution of hyperfine fields, for the density distribution of isomer shifts with lines in the Lorentz or Gauss forms [2], and for filtering and reducing noises [3]. Each of these methods is useful but has a limited range of application. In fact, the previously known algorithms did not react to hardly noticeable primary features of the spectrum, but identify the dominant components only. This is caused by the least squares methods which are applied to the whole experimental spectrum without any analysis of its details. In the presented approach, the difference between the experimental spectrum and the known part of the spectral structure defines the next Lorentzian. This method is effective for isolation of fine details of the spectrum, although it requires (see [4]) a well-elaborated algorithmic procedure presented in this paper.

1. Results and discussion

Below we give a detailed description and a proof of the new algorithm for the Mössbauer spectra analysis. Let

$$\varphi_s(x) = \frac{A_s b_s^2}{b_s^2 + (x - x_s)^2},$$

$$\varphi(x) = \varphi(x, b) = \frac{b}{b^2 + x^2}, \tag{2}$$

and let

$$\Phi_s(t) = [\varphi * \varphi_s]$$

be the convolution of the functions $\varphi_s(x)$ and $\varphi(x)$. Then

$$\Phi_s(t) = \Phi_s(t, b) = \frac{A_s b_s \pi (b + b_s)}{(b + b_s)^2 + (t - x_s)^2}, \quad s = 1, 2, \ldots, n \tag{3}$$

and

$$\max_t \Big| \frac{\partial^r \Phi_s(t, b)}{\partial b^r} \Big| = \frac{\pi A_s b_s r!}{(b + b_s)^{r+1}} \quad s = 1, 2, \ldots, n, \quad r \in \mathbb{N}, \tag{4}$$

where the maximum is attained for $t = x_s$ only. Equality (4) follows easily from the following formula:

$$\Big(\frac{\partial}{\partial p}\Big)^r \Big(\frac{p}{p^2 + \xi^2}\Big) = \Big(\frac{\partial}{\partial p}\Big)^r \Big(\frac{1}{2(p - i\xi)} + \frac{1}{2(p + i\xi)}\Big) = \frac{(-1)^r}{2}\Big(\frac{r!}{(p - i\xi)^{r+1}} + \frac{r!}{(p + i\xi)^{r+1}}\Big).$$

Indeed, this equality implies the relations

$$\Big|\Big(\frac{\partial}{\partial p}\Big)^r \Big(\frac{p}{p^2 + \xi^2}\Big)\Big| = \Big|\frac{(-1)^r}{2}\Big(\frac{r!}{(p - i\xi)^{r+1}} + \frac{r!}{(p + i\xi)^{r+1}}\Big)\Big|$$

$$= \Big|\frac{(-1)^r (r)!}{2}\Big(\frac{1}{\rho^{r+1}\exp(-i\varphi(r + 1))} + \frac{1}{\rho^{r+1}\exp(i\varphi(r + 1))}\Big)\Big|$$

A NEW ALGORITHM FOR ANALYSIS OF EXPERIMENTAL MÖSSBAUER SPECTRA

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Abstract: A new approach to analyze the nuclear gamma resonance (NGR) spectra is presented and justified in the paper. The algorithm successively spots the Lorentz lines in the experimental spectrum by a certain optimization procedures. In Mössbauer spectroscopy, the primary analysis is based on the representation of the transmission integral of an experimental spectrum by the sum of Lorentzians. In the general case, a number of lines and values of parameters in Lorentzians are unknown. The problem is to find them. In practice, before the experimental data processing, one elaborates a model of the Mössbauer spectrum. Such a model is usually based on some additional information. Taking into account physical restrictions, one forms the shape of the lines which are close to the normalized experimental Mössbauer spectrum. This is done by choosing the remaining free parameters of the model. However, this approach does not guarantee a proper model. A reasonable way to construct a structural NGR spectrum decomposition should be based on its model-free analysis. Some model-free methods of the NGR spectra analysis have been implemented in a number of known algorithms. Each of these methods is useful but has a limited range of application. In fact, the previously known algorithms did not react to hardly noticeable primary features of the experimental spectrum, but identify the dominant components only. In the proposed approach, the difference between the experimental spectrum and the known already determined part of the spectral structure defines the next Lorentzian. This method is effective for isolation of fine details of the spectrum, although it requires a well-elaborated algorithmic procedure presented in this paper.

Key words: Nuclear gamma resonance (NGR) spectra.

Introduction

In Mössbauer spectroscopy, the primary analysis of the NGR spectrum structure is based on the representation of the transmission integral of an experimental spectrum $f(x)$ by the following sum of Lorentzians:

$$f(x)=\sum_{s=1}^{n}\frac{A_s}{1+\Big(\dfrac{x-x_s}{b_s}\Big)^2},\quad A_s>0,\quad b_s>0,\quad x\in\mathbb{R}, \tag{1}$$

where x_s is the position of the maximum of the sth Lorentz line on the velocity scale, and $2b_s$ is its width. In the general case, the number n of lines and the values of parameters A_s, x_s, and b_s are unknown. The problem is to find them for a given function $f(x)$. The Mössbauer spectrum is always measured in some bounded velocity range. The maximal number of lines in the spectrum is limited by the number of probable positions of the Mössbauer atom in the crystal lattice and by the nature of changes of the nuclear energy levels (e.g., isomer shift, quadropole splitting, or hyperfine splitting).

For most of metallic alloys and oxide compounds, the evaluation of the structural parameters of the Mössbauer spectrum is a mathematically difficult nonlinear problem.

Corollary 1. *Let z be an end point of the interval $\mathbb{I}$, $1 \le q < \infty$, and $n \ge 1$. The polynomial ϱ_n^* of degree n with the unit leading coefficient that deviates the least from zero in the space L_q^w with weight* (2.11) *is the unique extremal polynomial in inequality* (2.7).

Special cases of this statement are given in [1, Theorem 1], [2, Theorem 2], [3, Theorem 2], [4, Theorem 3]; they have been proved there by means of other arguments.

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REFERENCES

1. **Arestov V.V., Deikalova M.V.** Nikol'skii inequality for algebraic polynomials on a multidimensional Euclidean sphere // Proc. Steklov Inst. Math., 2014. Vol. 284. Suppl. 1. P. S9–S23. DOI: 10.1134/S0081543814020023
2. **Arestov V., Deikalova M.** Nikol'skii inequality between the uniform norm and L_q-norm with ultraspherical weight of algebraic polynomials on an interval // Comput. Methods Funct. Theory, 2015. Vol. 15, no. 4. P. 689–708. DOI: 10.1007/s40315-015-0134-y
3. **Arestov V., Deikalova M.** Nikol'skii inequality between the uniform norm and L_q-norm with Jacobi weight of algebraic polynomials on an interval // Analysis Math., 2016. Vol. 42, no. 2. P. 91–120. DOI: 10.1007/s10476-016-0201-2
4. **Arestov V., Deikalova M., Horváth Á.** On Nikol'skii type inequality between the uniform norm and the integral q-norm with Laguerre weight of algebraic polynomials on the half-line // J. Approx. Theory, 2017. Vol. 222. P. 40–54. DOI: 10.1016/j.jat.2017.05.005
5. **Babenko V.F., Korneichuk N.P., Ligun A.A.** Extremal properties of polynomials and splines. New York: Nova Science, 1996.
6. **Day M.M.** Normed linear space. Berlin; Göttingen; Heidelberg: Springer, 1958.
7. **Diestel J.** Geometry of Banach spaces: selected topics. Berlin: Springer, 1975.
8. **Dunford N., Schwartz J.** Linear operators: general theory. New York: Interscience, 1958.
9. **Gol'shtein E.G.** Duality theory in mathematical programming and its applications. Moscow: Nauka, 1971. 351 p. [in Russian].
10. Handbook of the Geometry of Banach Spaces. Ed. W.B. Johnson and J. Lindenstrauss. Elsevier, 2001. Vol. 1.
11. **James R.** Characterizations of reflexivity // Studia Mathematica, 1964. Vol. 23, iss. 3. P. 205–216.
12. **Kolmogorov A.N., Fomin S.V.** Elements of the theory of functions and functional analysis. Moscow: Fizmatlit, 2004.
13. **Korneichuk N.P.** Extremal problems of approximation theory. Moscow: Nauka, 1976. [in Russian].
14. **Milovanović G.V., Mitrinović D.S., Rassias Th.M.** Topics in polynomials: extremal problems, inequalities, zeros. Singapore: World Scientific, 1994. 821 p.
15. **Nikol'skii S.M.** Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables // Trudy Mat. Inst. Steklova, 1951. Vol. 38. P. 244–278 [in Russian].
16. **Simonov I.E., Glazyrina P.Yu.** Sharp Markov–Nikol'skii inequality with respect to the uniform norm and the integral norm with Chebyshev weight // J. Approx. Theory, 2015. Vol. 192. P. 69–81. DOI: 10.1016/j.jat.2014.10.009
17. **Singer I.** Best approximation in normed linear spaces by elements of linear subspaces. Berlin: Springer, 1970.
18. **Szegő G., Zygmund A.** On certain mean values of polynomials // J. Anal. Math., 1953. Vol. 3, no. 1. P. 225–244.

Finally, let us check that the exact degree of the polynomial ϱ_n is n or $n-1$. Indeed, if ϱ_n has degree at most $n-2$, then the polynomial $p_{n-1}(x) = (x-z)\varrho_n(x)$ has degree at most $n-1$. The integral on the left-hand side of (2.9) is positive for this polynomial. This contradicts property (2.9). The theorem is proved. $\square$

Example. Consider the special case of problem (2.7) in the space $L = L(-1,1)$ of functions that are integrable over the interval $I = [-1,1]$ with the unit weight, with $n = 1$ and $z = 0$. In other words, we are interested in the sharp inequality

$$|p(0)| \le D\|p\|_L, \quad p \in \mathscr{P}_1. \tag{2.10}$$

It is easy to verify that we have the formula

$$p(0) = \frac{1}{2}\int_{-1}^{1} p(t)dt, \quad p \in \mathscr{P}_1.$$

Using this formula, it is straightforward that the best constant in inequality (2.10) is $D = 1/2$ and that every polynomial of constant sign on $(-1,1)$ is extremal. Thus, an extremal polynomial in inequality (2.7) may be not unique, may have (real) zeros outside the interval $\mathbb{I}$, and may have the exact degree $n-1$.

For an end point z of the interval $\mathbb{I}$, we are able to derive more information about the properties of extremal polynomials in inequality (2.7) from Theorem 4. In this case, the product $(x-z)\upsilon(x)$ on the left-hand side of (2.9) has constant sign on $\mathbb{I}$. Therefore, using property (2.9), it is not difficult to see that an extremal polynomial ϱ_n has degree exactly n, all n zeros of this polynomial are simple and lie in the interior of the interval $\mathbb{I}$. Property (2.9) implies also that the extremal polynomial ϱ_n is unique for all $1 \le q < \infty$. Indeed, let ϱ_n and η_n be two polynomials that solve problem (2.8). The same property is true for their half-sum $(\varrho_n + \eta_n)/2$; therefore, we have $\|\varrho_n + \eta_n\|_{L_q^\upsilon} = \|\varrho_n\|_{L_q^\upsilon} + \|\eta_n\|_{L_q^\upsilon}$. For $1 < q < \infty$, it follows immediately that $\eta_n = \varrho_n$. For $q = 1$, it only follows that the polynomials η_n and ϱ_n have the same sign almost everywhere on $\mathbb{I}$. But the zeros of these polynomials are simple and lie in the interior of the interval $\mathbb{I}$; therefore, the polynomials η_n and ϱ_n have the same set of zeros and, hence, it follows that these polynomials coincide in the case when $q = 1$, too.

For a given weight υ and a given point $z \in \mathbb{I}$, we define the weight

$$w(x) = |x-z|\,\upsilon(x) \tag{2.11}$$

on the interval $\mathbb{I}$. We denote by $\varrho_n^* = \varrho_{n,w,q}^*$ the polynomial of degree $n \ge 1$ with the unit leading coefficient that deviates the least from zero in the space $L_q^w = L_q^w(\mathbb{I})$, i.e., is a solution of the problem

$$\min\{\|p_n\|_{L_q^w} : p_n \in \mathscr{P}_n^1\} = \|\varrho_n^*\|_{L_q^w}$$

on the set $\mathscr{P}_n^1$ of polynomials of degree n with the leading coefficient equal to 1.

The polynomial ϱ_n^* can be characterized by the property that the function $|\varrho_n^*|^{q-1}\operatorname{sign}\varrho_n^*$ is orthogonal to the space $\mathscr{P}_{n-1}$ (see, for example, [13, Ch. 3, Sect. 3.3, Theorems 3.3.1, 3.3.2]), i.e.,

$$\int_{\mathbb{I}} w(x)\,p_{n-1}(x)|\varrho_n^*(x)|^{q-1}\operatorname{sign}\varrho_n^*(x)\,dx = 0, \quad p_{n-1} \in \mathscr{P}_{n-1}.$$

This property coincides with property (2.9). Therefore, the polynomials ϱ_n and ϱ_n^* differ only by a constant factor. Thus, the following statement holds.

with the smallest possible constant $D_n[z] = D(n, \upsilon, q; z)$ for points $z \in \mathbb{I}$. Such inequalities are of independent interest, but they are also important in connection with inequality (2.6) since

$$M(n) = \sup\{D_n[z]\, u(z) \colon z \in \mathbb{I}\}.$$

In a number of important cases, the product $D_n[z]\, u(z)$ takes its maximal value with respect to $z \in \mathbb{I}$ at an end point of the interval $\mathbb{I}$; see, e.g., [2–4, 16] and the references therein.

In the setup we consider in this section, (1.6) and (1.3) take the form

$$\Delta_n[z] = \inf\{\|p_n\|_{L_q^\upsilon(\mathbb{I})} \colon p_n \in \mathscr{P}_n[z]\}, \tag{2.8}$$

$$\mathscr{P}_n[z] = \{p_n \in \mathscr{P}_n \colon p_n(z) = 1\}.$$

Theorem 4. *For $1 \le q < \infty$, the following is true for an extremal polynomial in inequality* (2.7)*.*

(1) *An extremal polynomial ϱ_n in inequality* (2.7) *exists, it has real coefficients, all its roots are real, and its degree is at least $n-1$. In the case when $1 < q < \infty$, the extremal polynomial is unique.*

(2) *A polynomial $\varrho_n \in \mathscr{P}_n$ is extremal in inequality* (2.7) *if and only if*

$$\int_{\mathbb{I}} p_{n-1}(x)(x-z)\upsilon(x)|\varrho_n(x)|^{q-1}\mathrm{sign}\, \varrho_n(x)dx = 0 \quad \textit{for all} \quad p_{n-1} \in \mathscr{P}_{n-1}. \tag{2.9}$$

P r o o f. Inequality (2.7) is a special case of inequality (2.2) for the functional $\psi(p) = p(z)$, $p \in \mathscr{P}_n$. In this case, set (2.3) is formed by polynomials of the form $s(x) = (x-z)p_{n-1}(x)$, $p_{n-1} \in \mathscr{P}_{n-1}$. Therefore, condition (2.4) for an extremal polynomial ϱ_n in inequality (2.7) takes the form (2.9). Thus, the second statement of Theorem 4 is proved. Without loss of generality, we may assume that $\varrho_n(z) = 1$; for, consider $\varrho_n/\varrho_n(z)$ instead of the polynomial ϱ_n, if necessary.

The polynomial ϱ_n is also a solution of problem (2.8). We will study some properties of the polynomial ϱ_n using this fact. In general, the coefficients $\{c_k\}_{k=0}^n$ of the polynomial ϱ_n are complex, namely, $c_k = a_k + ib_k$, $a_k, b_k \in \mathbb{R}$. We write ϱ_n in the form $\varrho_n = u_n + iv_n$, where

$$u_n(x) = (\mathrm{Re}\, \varrho_n)(x) = \sum_{k=0}^n a_k x^k, \quad v_n(x) = (\mathrm{Im}\, \varrho_n)(x) = \sum_{k=0}^n b_k x^k$$

are real polynomials (on $\mathbb{R}$). Obviously, $u_n(z) = \varrho_n(z) = 1$; hence, $u_n \in \mathscr{P}_n[z]$. If $b_k \neq 0$ for at least one k, $0 \le k \le n$, then the strict inequality $|u_n(x)| < |\varrho_n(x)|$ holds for all $x \in \mathbb{I}$ except for zeros of the polynomial v_n. Consequently, the strict inequality $\|u_n\|_{L_q^\upsilon(\mathbb{I})} < \|\varrho_n n\|_{L_q^\upsilon(\mathbb{I})}$ holds for the norms of these polynomials. The latter is a contradiction to the fact that the polynomial ϱ_n is extremal in (2.8). This proves that the coefficients of the polynomial ϱ_n are real.

Assume that the polynomial ϱ_n has a zero ζ which is not real. Since the polynomial ϱ_n is real, we conclude that $\overline{\zeta}$ is also a zero of ϱ_n. Consequently, $\varrho_n(x) = q_{n-2}(x)|x-\zeta|^2$, where q_{n-2} is a polynomial of degree at most $n-2$. The polynomial $p_{n-1}(x) = q_{n-2}(x)(x-z)$ has degree at most $n-1$. The left-hand side of (2.9) is positive for this polynomial:

$$\int_{\mathbb{I}} p_{n-1}(x)(x-z)\upsilon(x)|\varrho_n(x)|^{q-1}\mathrm{sign}\, \varrho_n(x)dx =$$
$$= \int_{\mathbb{I}} (x-z)^2\upsilon(x)|q_{n-2}(x)|^q|x-\zeta|^{2(q-1)}\mathrm{sign}\, q_{n-2}(x)dx > 0.$$

This contradicts property (2.9). Thus, the polynomial ϱ_n can have only real zeros.

For a pair of functions $\phi \in L_\infty(\mathbb{I})$ and $f \in L_1^\upsilon(\mathbb{I})$, the inequality

$$\left| \int_{\mathbb{I}} f(t)\overline{\phi}(t)\upsilon(t)dt \right| \le \|\phi\|_{L_\infty(\mathbb{I})} \|f\|_{L_1^\upsilon(\mathbb{I})}$$

turns into an equality if and only if the following three conditions hold:

(1) the set

$$\mathbb{I}(\phi) = \{t \in \mathbb{I} : \ |\phi(t)| = \|\phi\|_{L_\infty(\mathbb{I})}\}$$

where the absolute value of the function ϕ takes its maximum has a positive measure;

(2) the function f vanishes almost everywhere outside the set $\mathbb{I}(\phi)$;

(3) the product $f\overline{\phi}$ has the same sign almost everywhere on the set

$$\Theta f = \{t \in \mathbb{I} \colon f(t) \neq 0\}.$$

Taking into account these observations, it is not difficult to conclude that a supporting functional of a function $f \in S(L_1^\upsilon(\mathbb{I}))$ has the form (2.5), where the function ϕ satisfies the following conditions: $\phi = \operatorname{sign} f$ almost everywhere on Θf and $|\phi| \le 1$ almost everywhere outside Θf.

Consequently, a function $f \in S(L_1^\upsilon(\mathbb{I}))$ is a smooth point of the unit sphere of the space $L_1^\upsilon(\mathbb{I})$ if and only if f is nonzero almost everywhere on $\mathbb{I}$; the supporting functional in this case has the form (2.5) with the function $\phi = \operatorname{sign} f$. In particular, this property holds in the case if f is an algebraic polynomial. Thus, under the assumptions of Theorem 3 for $q = 1$, property (Γ) holds.

For $1 < q < \infty$, the dual space of $L_q = L_q^\upsilon(\mathbb{I})$ is $L_{q'} = L_{q'}^\upsilon(\mathbb{I})$, $1/q + 1/q' = 1$. The space $L_{q'}^\upsilon(\mathbb{I})$ with $1 < q' < \infty$ is uniformly convex; hence, the space $L_q^\upsilon(\mathbb{I})$ is smooth.

Thus, we have shown that all assumptions of Theorem 1 are fulfilled under the assumptions of Theorem 3. Thus, also the statement of Theorem 1 holds. For $1 < q < \infty$, the space $L_q^\upsilon(\mathbb{I})$ is uniformly smooth, hence, the extremal polynomial in inequality (2.2) is unique. This proves the theorem. ☐

2.2. Pointwise Nikol'skii inequality for algebraic polynomials on an interval

Let u be another, this time continuous weight on $\mathbb{I}$. Along with $L_q^\upsilon(\mathbb{I})$, we consider the space $C = C(\mathbb{I}, u)$ of complex-valued continuous functions f such that the product fu is bounded on $\mathbb{I}$, endowed the (uniform weighted) norm

$$\|f\|_{C(\mathbb{I},u)} = \sup\{|f(x)u(x)| \colon x \in \mathbb{I}\}.$$

We will assume that $\mathscr{P}_n$ is contained not only in $L_q^\upsilon(\mathbb{I})$ but also in $C(\mathbb{I}, u)$; the latter is equivalent to the fact that the function $u(x)(1 + |x|^n)$ is bounded on $\mathbb{I}$.

Denote by $M(n) = M(n, u, \upsilon)_q$ the best (the smallest possible) constant in the inequality

$$\|p\|_{C(\mathbb{I},u)} \le M(n)\, \|p\|_{L_q^\upsilon(\mathbb{I})}, \quad p \in \mathscr{P}_n, \tag{2.6}$$

on the set $\mathscr{P}_n$. Inequality (2.6) is a special case of an inequality between different metrics, or the Nikol'skii inequality. Such inequalities appeared for the first time in Nikol'skii's paper [15] and, shortly after that, in a paper by Szegő and Zygmund [18]. Similar inequalities and, more generally, inequalities between the uniform norm and weighted integral norms of algebraic and trigonometric polynomials and their derivatives have been studied over a period of more than 150 years, starting with the works of Chebyshev and his students—the Markov brothers. Further information and references on this topic can be found, e.g., in monographs [5, 14] and papers [2, 3, 16].

Along with (2.6), we consider the pointwise inequality

$$|p_n(z)| \le D_n[z]\, \|p_n\|_{L_q^\upsilon(\mathbb{I})}, \quad p_n \in \mathscr{P}_n, \tag{2.7}$$

For $q = \infty$, we assume that $L^\upsilon_\infty(\mathbb{I})$ is the space $L_\infty = L_\infty(\mathbb{I})$ of essentially bounded functions on $\mathbb{I}$ with the norm

$$\|f\|_{L_\infty(\mathbb{I})} = \operatorname{ess\,sup}\{|f(t)| \colon t \in \mathbb{I}\}.$$

Let $\mathscr{P}_n = \mathscr{P}_n(\mathbb{C})$ for $n \geq 0$ be the set of algebraic polynomials (in one variable) of degree at most n with complex coefficients. We will assume that $\mathscr{P}_n \subset L^\upsilon_q(\mathbb{I})$; this condition is equivalent to the fact that the function $1 + |x|^n$ belongs to the space $L^\upsilon_q(\mathbb{I})$.

2.1. Arbitrary bounded linear functionals on the space of algebraic polynomials

Assume that ψ is a linear functional on $\mathscr{P}_n$. Since $\mathscr{P}_n$ is finite-dimensional, the functional ψ on $\mathscr{P}_n$ is bounded and its norm

$$D(\psi; \mathscr{P}_n)_q = \sup\{|\psi(p)| \colon p \in \mathscr{P}_n,\ \|p\|_{L^\upsilon_q(\mathbb{I})} = 1\} \tag{2.1}$$

is attained at a certain polynomial $\varrho_n = \varrho_{\psi,\mathscr{P}_n,q} \in \mathscr{P}_n$ with the property

$$\|\varrho_{\psi,\mathscr{P}_n,q}\|_{L_q(\mathbb{I},\upsilon)} = 1.$$

In the study of extremal problems for polynomials, it is an important fact that the value $D_n(\psi) = D(\psi; \mathscr{P}_n)_q$ is the smallest possible (the best) constant in the inequality

$$|\psi(p)| \leq D_n(\psi)\|p\|_{L^\upsilon_q(\mathbb{I})}, \quad p \in \mathscr{P}_n. \tag{2.2}$$

Inequality (2.2) turns into an equality at the polynomial ϱ_n, i.e., ϱ_n is extremal in (2.2). It is clear that the polynomial $c\varrho_n$ with an arbitrary constant $c \in \mathbb{C}$ is also extremal in (2.2). If all extremal polynomials in inequality (2.2) have the form $c\varrho_n,\ c \in \mathbb{C}$, we say that ϱ_n is the *unique* extremal polynomial of inequality (2.2) (or of problem (2.1)). In what follows, we assume that $\psi \not\equiv 0$; this is equivalent to the fact that $|\psi(\varrho_n)| = D(\psi; \mathscr{P}_n) > 0$.

Consider the annihilator

$$\mathscr{P}_n(\psi) = \{p \in \mathscr{P}_n \colon \psi(p) = 0\} \tag{2.3}$$

of the functional ψ in the set $\mathscr{P}_n$. This set is a subspace of $\mathscr{P}_n$ of codimension 1. This subspace is formed by polynomials of the form

$$s = p - \frac{\psi(p)}{\psi(\varrho_n)}\varrho_n, \quad p \in \mathscr{P}_n.$$

Theorem 3. *Let $1 \leq q < \infty$. A polynomial $\varrho_n = \varrho_{\psi,\mathscr{P}_n,q} \in \mathscr{P}_n$ which is extremal in inequality* (2.2) *exists. A polynomial $\varrho_n \in \mathscr{P}_n$ is extremal if and only if*

$$\int_{\mathbb{I}} s(x)\upsilon(x)|\varrho_n(x)|^{q-1}\operatorname{sign}\varrho_n(x)dx = 0 \quad \text{for all} \quad s \in \mathscr{P}_n(\psi). \tag{2.4}$$

In the case when $1 < q < \infty$, this extremal polynomial is unique (*up to a constant factor*).

P r o o f. We check that all assumptions of Theorem 1 are fulfilled under the assumptions of Theorem 3. The set $\mathscr{P}_n = \mathscr{P}_n(\mathbb{C})$ of algebraic polynomials of degree at most n is a finite-dimensional subspace of $L^\upsilon_q(\mathbb{I})$. This guarantees that property (R) holds.

Now let us verify property (Γ). We start with the case $q = 1$. The dual space of $L = L^\upsilon_1(\mathbb{I})$ is the space $L_\infty = L_\infty(\mathbb{I})$ of essentially bounded functions on $\mathbb{I}$. A functional $\Phi \in X^*$ has the representation

$$\Phi(f) = \int_{\mathbb{I}} f(t)\overline{\phi}(t)\upsilon(t)dt, \quad f \in L^\upsilon_1(\mathbb{I}), \tag{2.5}$$

where $\phi \in L_\infty(\mathbb{I})$ and $\|\Phi\|_{L^*} = \|\phi\|_{L_\infty(\mathbb{I})}$.

P r o o f. The functional ψ is a bounded linear functional on the space P endowed with the norm $\|\cdot\|_X$, and the norm of this functional on P is equal to (1.1). By the Hahn–Banach theorem (cf. [8, Ch. II, Sect. 3, Theorem 11] or [12, Ch. III, Sect. 5.4]), the functional ψ can be extended to a functional on the whole space X with the same norm; we denote this extension by Ψ.

Since the functional Ψ is an extension of the functional ψ from P to X with the same norm, the norm of the functional Ψ in the space X is attained at an extremal element $\varrho \in P$ of problem (1.1). By property (Γ), the functional Ψ differs from the functional $F = F[\varrho] \in X^*$ only by a constant factor $\gamma(P)$:

$$\Psi(p) = \gamma(P)F[\varrho](p), \quad p \in X.$$

In particular, (1.12) holds. Taking $p = \varrho$ in (1.12), we see that $|\gamma(P)| = D(\psi, P)$. This proves representation (1.12). □

Lemma 4. *Assume that a Banach space X and its subspace P satisfy properties* (R) *and* (Γ). *If an element $\varrho \in P$, $\varrho \not\equiv 0$, or, more precisely, the supporting functional $F = F[\varrho] \in S(X^*)$ at the element ϱ has property* (1.9), *then ϱ is an extremal element of problem* (1.1).

P r o o f. Assume that an element $\varrho \in P$, $\varrho \not\equiv 0$, has property (1.9); without loss of generality, we may assume that $\|\varrho\|_X = 1$. We consider the linear functional on the set P defined by the formula

$$\Psi_0(p) = F[\varrho](p). \tag{1.13}$$

For any $p \in P$, the element $s = \psi(\varrho)p - \psi(p)\varrho$ belongs to the set $P(\psi)$. Due to (1.9), we have

$$\psi(\varrho)\Psi_0(p) - \psi(p)\Psi_0(\varrho) = 0. \tag{1.14}$$

By (1.8) and (1.13), we have $\Psi_0(\varrho) = F[\varrho](p) = 1$. Thus, (1.14) can be rewritten as

$$\psi(p) = \psi(\varrho)\,\Psi_0(p), \quad p \in P. \tag{1.15}$$

We conclude that

$$|\psi(p)| = |\psi(\varrho)|\,|\Psi_0(p)| \le |\psi(\varrho)|\,\|p\|.$$

Consequently, $D(\psi, P) \le |\psi(\varrho)|$. Since $\|\varrho\|_X = 1$, we have $D(\psi, P) \ge |\psi(\varrho)|$. It follows that $D(\psi, P) = |\psi(\varrho)|$ and the element ϱ is extremal in problem (1.1). □

1.4.2. Proof of Theorem 1

Formula (1.12) implies that an extremal element of problem (1.1) has property (1.9). According to Lemma 4, the inverse statement holds. This proves Theorem 1. □

2. Bounded linear functionals on the set of algebraic polynomials in spaces L_q^υ, $1 \le q < \infty$

Assume that $\mathbb{I}$ is a finite or infinite closed interval of the real line and υ is a nonnegative function that is integrable and almost everywhere nonzero on $\mathbb{I}$; we will call such functions weights on $\mathbb{I}$. Denote by $L_q = L_q^\upsilon(\mathbb{I})$, $1 \le q < \infty$, the space of (complex-valued) measurable functions f on $\mathbb{I}$ such that the product $|f|^q\upsilon$ is integrable on $\mathbb{I}$; this is a Banach space with the norm

$$\|f\|_{L_q^\upsilon(\mathbb{I})} = \left(\int_{\mathbb{I}} |f(x)|^q \upsilon(x)\,dx\right)^{1/q}, \quad f \in L_q^\upsilon(\mathbb{I}).$$

Lemma 2. *A complex functional $F \in X^*_{\mathbb{C}}$ attains its norm at a point $x \in S(X)$ and $F(x) > 0$ if and only if its real part $f = \operatorname{Re} F$ has the same properties.*

P r o o f. Suppose F is as described in the lemma. By (1.8) and (1.11), we have

$$\|F\|_{X^*_{\mathbb{C}}} = \|f\|_{X^*_{\mathbb{R}}} = F(x) = f(x).$$

Consequently, f has the same properties as F. Conversely, suppose f has the described properties. Then, by (1.11), we have

$$f(x) \leq \sqrt{(f(x))^2 + (f(ix))^2} = |F(x)| \leq \|F\|_{X^*_{\mathbb{C}}} = \|f\|_{X^*_{\mathbb{R}}}.$$

Consequently, $f(x){=}F(x){=}\|F\|_{X^*_{\mathbb{C}}}$; hence, $F(x){>}0$. Thus, F has the described properties, too. □

As we have mentioned above, a functional $F \in X^*$ in a complex Banach space is called a supporting functional at a point x (to the sphere $S_{\|x\|}(X)$ of radius $\|x\|$ with center at 0) if its real part $\operatorname{Re} F$ is a (real) supporting functional, cf. [8, Ch. V, Sect. 9.4]. Due to Lemma 2, the N-property of the functional $F \in X^*$ at a point x is equivalent to the property that $F \in X^*$ is a supporting functional at this point.

Theorem 2. *Assume that the space $X^* = X^*_{\mathbb{C}}$ of complex bounded linear functionals in a complex Banach space X is strictly convex. Then X is smooth.*

P r o o f. Recall that a Banach space is called strongly convex if its unit sphere does not contain any non-degenerate segments, see, e.g. [8, Ch. V, Sect. 11.7]. As we have mentioned above, the statement of the theorem is well-known for real Banach spaces, cf. [7, Ch. I, Sect. 2, Theorems 1 and 2], [6, Ch. VII, Sect. 2].

Using Lemma 1, it is not difficult to see that $X^*_{\mathbb{C}}$ is strongly convex if and only if $X^*_{\mathbb{R}}$ is. Thus, under the assumptions of the theorem, the space $X_{\mathbb{R}}$ is smooth. This means that, at any point $x \in S(X)$, there is only one real bounded linear functional f whose norm is equal to 1 and is attained at x, with $f(x) > 0$. By Lemma 1, this implies that, at every point $x \in S(X)$, there is only one functional $F \in X^*_{\mathbb{C}}$ with the unit norm and with the N-property at the point x. But this means that the space $X = X_{\mathbb{C}}$ is smooth. □

1.4. Proof of Theorem 1

Theorem 1 follows from the two auxiliary statements proved below. In what follows, we will suppose without loss of generality that all supporting functionals $F = F[x]$ at points $x \in X$, $x \neq 0$, have the norm $\|F\|_{X^*} = 1$.

1.4.1. Auxiliary statements

Lemma 3. *Assume that a Banach space X and its subspace P satisfy properties* (R) *and* (Γ). *Let $\varrho \in P$, $\varrho \not\equiv 0$, be an extremal element of problem* (1.1), *and let $F = F[\varrho] \in S(X^*)$ be the supporting functional at the element $\varrho \in P$. Then the following representation holds*:

$$\psi(p) = \gamma(P) F[\varrho](p), \quad p \in P, \tag{1.12}$$

where $\gamma(P)$ is a constant with the property $|\gamma(P)| = D(\psi, P)$.

the strict convexity of the dual space is a sufficient condition for the smoothness of the original space. The inverse statement does not hold in the general case. The smoothness of a space implies the strict convexity of the dual space only for reflexive spaces. Details on these topics can be found, e.g., in [7, Ch. I, Sect. 2, Theorems 1, 2] and [6, Ch. VII, Sect. 2]. In this section, we discuss smoothness in complex Banach spaces. The author neither claims that the results are novel nor that the ideas are original.

Let $X = X_{\mathbb{C}}$ be a complex Banach space. We also may consider this space as a real Banach space $X = X_{\mathbb{R}}$, i.e., a linear space over the field $\mathbb{R}$ of real numbers. Let $X^*_{\mathbb{R}}$ be the corresponding dual (real) Banach space, i.e., the space of real-valued bounded linear (over the field $\mathbb{R}$ of real numbers) functionals on $X_{\mathbb{R}}$.

The following statement is not new, cf. [8, Ch. II, Sect. 3, Theorem 11] or [9, Ch. 10, Sect. 1, Lemma 1.1]. We will give it here in the form we need in what follows. Moreover, it is useful for our purposes to give a proof of this statement.

Lemma 1. *The formula*

$$F(x) = f(x) - if(ix), \quad x \in X, \tag{1.10}$$

where $F \in X^*_{\mathbb{C}}$ and $f \in X^*_{\mathbb{R}}$, sets a one-to-one correspondence between the spaces $X^*_{\mathbb{C}}$ and $X^*_{\mathbb{R}}$. Moreover, mapping (1.10) *is an isometry, i.e.,*

$$\|F\|_{X^*_{\mathbb{C}}} = \|f\|_{X^*_{\mathbb{R}}}. \tag{1.11}$$

P r o o f. For a complex functional $F \in X^*_{\mathbb{C}}$, we consider its real part $f = \operatorname{Re} F$; it is a functional from $X^*_{\mathbb{R}}$. The functional F is uniquely determined by $f = \operatorname{Re} F$ by means of formula (1.10). Indeed, define $g = -\operatorname{Im} F$, then $F(x) = f(x) - ig(x),\ x \in X$. By the (complex) homogeneity of the functional F, we have $F(x) = -iF(ix) = -if(ix) + g(ix),\ x \in X$. Consequently, $g(x) = f(ix)$, which proves representation (1.10).

Conversely, let $f \in X^*_{\mathbb{R}}$. Consider a (complex) functional F given by formula (1.10). Obviously, F is additive. Next we will show that it is (complex) homogeneous. For a point $x \in X$ and a number $\zeta = \alpha + i\beta \in \mathbb{C}$, we have

$$F(\zeta x) = F((\alpha + i\beta)x) = f((\alpha + i\beta)x) - if(i(\alpha + i\beta)x) =$$

$$= \alpha f(x) + \beta f(ix) - i\alpha f(ix) + i\beta f(x) = (\alpha + i\beta)f(x) + (\beta - i\alpha)f(ix) =$$

$$= (\alpha + i\beta)(f(x) - if(ix)) = \zeta F(x).$$

Thus, we see that functional (1.10) is homogeneous.

It follows that formula (1.10) sets a one-to-one correspondence between the complex and the real dual spaces $X^*_{\mathbb{C}}$ and $X^*_{\mathbb{R}}$, respectively.

Now we show that (1.10) is an isometry, i.e., property (1.11) holds. The inequality $\|f\|_{X^*_{\mathbb{R}}} \le \|F\|_{X^*_{\mathbb{C}}}$ is obvious. Further on, for an arbitrary point $x \in X$ and real θ, we have

$$e^{i\theta} F(x) = F\left(e^{i\theta} x\right) = f\left(e^{i\theta} x\right) - if\left(ie^{i\theta} x\right).$$

In particular, for $\theta = -\arg(F(x))$, the latter equality takes the form

$$|F(x)| = f\left(e^{i\theta} x\right) - if\left(ie^{i\theta} x\right) = f\left(e^{i\theta} x\right).$$

Consequently, $|F(x)| \le \|f\|_{X^*_{\mathbb{R}}} \|x\|,\ x \in X$, and therefore the estimate $\|F\|_{X^*_{\mathbb{C}}} \le \|f\|_{X^*_{\mathbb{R}}}$ holds. Thus, (1.11) holds. This proves the lemma. $\square$

All further statements in this section are in fact consequences of Lemma 1.

or [12, Ch. III, Sect. 5.4]), a functional with this property always exists. However, it may be not unique. In a complex Banach space, a functional $F \in X^*$ is called a supporting functional at a point x (or, more precisely, a supporting or a tangent functional at a point x to the sphere $S_{\|x\|}(X)$ of radius $\|x\|$ with center at 0), if its real part $f = \operatorname{Re} F$ is a real supporting (tangent) functional, see, e.g., [8, Ch. V, Sect. 9.4]. Indeed, for a functional $F \in X^*$ in a complex Banach space, the properties that the functional possesses the N-property at a point x and that its real part is a supporting functional are equivalent; we will discuss this below in Section 1.3. Starting from this point, we will interpret the $N[x]$-property of a functional $F \in X^*$ as a property of the functional $F \in X^*$ to be a supporting functional at the point x.

A point $x \in S(X)$ is called a smooth point of the sphere $S(X)$ if there exists only one supporting functional at x. If every point of the unit sphere of a space is a smooth point, then the space is called smooth. For details concerning smooth points of the unit sphere and, in general, of convex closed sets in real Banach spaces see, e.g., [7, Ch. I, Sect. 2, Theorems 1, 2] and [6, Ch. VII, Sect. 2]. The smoothness in complex Banach spaces has some special features; it will be discusses in Section 1.3 below.

The second assumption is the following one.

(Γ) *Assume that all points of the unit sphere $S(P) = S(X) \cap P$ of the subspace P are smooth points of the unit sphere $S(X)$ of the space X.*

Taking into account that problem (1.1) has the interpretation (1.4) in terms of approximations, one may expect the following result.

Theorem 1. *Assume that a Banach space X and its subspace P satisfy properties* (R) *and* (Γ). *Then the norm of a bounded linear functional ψ on P is attained at an element $\varrho \in S(P)$ if and only if the supporting functional $F = F[\varrho] \in X^*$ of the element $\varrho \in P$ vanishes on the set* (1.5), *i.e.,*

$$F[\varrho](s) = 0 \quad \text{for all} \quad s \in P(\psi). \tag{1.9}$$

Under the assumptions of the theorem, an extremal element with the property that the norm of the functional ψ on P is attained at it always exists but it is not necessarily unique; see the example after the proof of Theorem 4 in Section 2.2. To ensure the uniqueness of the extremal element, one needs additional restrictions on the problem. For example, if the space X is strictly normed then the extremal element is unique for every (bounded linear) functional on every subspace.

In the first section of the present paper, Theorem 1 will be proved and discussed. In the second section, Theorem 1 will be applied to obtain a corresponding statement for the pointwise inequality

$$|p_n(z)| \le D(z)\, \|p_n\|_{L_q^v(\mathbb{I})}, \quad p_n \in \mathscr{P}_n,$$

where $z \in \mathbb{I}$. In papers [1–4], extremal polynomials of inequality (1.7) (in the case when z is an end point of an interval) were characterized in terms that are formally different from those of Theorem 1. We will show in Section 2.2 that, in fact, Theorem 2 from [3] and its analogs from [1, 2, 4] follow from Theorem 1.

We will prove Theorem 1 using the natural arguments in terms of duality. However, the fact that X is a complex Banach space causes additional difficulties. In particular, one needs to first discuss the smoothness property for points of the unit sphere $S(X)$ of a (complex) Banach space X.

1.3. Smoothness in complex Banach spaces

In real Banach spaces, a smooth point of the unit sphere can be for example characterized by the fact that the norm of the space is Gateaux differentiable at this point. For a real Banach space, On the set

$$P[1](\psi) = \{p \in P\colon \psi(p) = 1\} \tag{1.3}$$

of elements of P where the functional ψ takes the value 1, we consider the value

$$\Delta(\psi; P) = \inf\{\|p\|_X : p \in P[1](\psi)\} \tag{1.4}$$

which is the least deviation of class (1.3) from zero in X. It is clear that $\Delta(\psi; P) = 1/D(\psi; P)$. Moreover, extremal elements in problem (1.4) and inequality (1.2) coincide. More precisely, each extremal element of problem (1.4) is extremal in (1.2); conversely, if ϱ is an extremal element of inequality (1.2), then $\varrho/\psi(\varrho)$ is extremal in (1.4). Thus, determining the sharp constant in inequality (1.2) is equivalent to determining the least deviation (1.4) of class (1.3) from zero.

Value (1.4) can be interpreted as the best approximation of an arbitrary element $\rho \in P[1](\psi)$ in the space X by the annihilator

$$P(\psi) = P[0](\psi) = \{p \in P : \ \psi(p) = 0\} \tag{1.5}$$

of the functional ψ in P, namely,

$$\Delta(\psi; P) = \inf\{\|\rho - p\|_X : p \in P(\psi)\}. \tag{1.6}$$

There is a rich theory developed to study problems of type (1.4) in real Banach spaces. This theory is based on arguments of duality; see, e.g., [13, Ch. 2]. In order to use this approach in the complex case, however, one needs in addition to discuss some questions of geometry of complex spaces.

In papers [1–4] coauthored by the author of the present paper, the authors studied the Nikol'skii inequality between the uniform norm of a polynomial and its norm in the space $L_q^v = L_q^v(\mathbb{I})$ with a weight v and $1 \le q < \infty$ on the set of algebraic polynomials $\mathscr{P}_n$ of degree at most $n \ge 1$ on a finite or infinite interval $\mathbb{I}$. One of the steps in these investigations was the study of the sharp inequality

$$|p_n(z_0)| \le D\, \|p_n\|_{L_q^v}, \quad p_n \in \mathscr{P}_n, \tag{1.7}$$

for an end point z_0 of the interval $\mathbb{I}$. Inequality (1.7) is a special case of (1.2) for $X = L_q^v(\mathbb{I})$, $P = \mathscr{P}_n$, and $\psi(p_n) = p_n(z_0)$. Results of [1–4] related to inequality (1.7) motivated the author to consider problem (1.2).

1.2. Main result

We will study problem (1.1) under the following two assumptions.

(R) *Assume that the norm of any bounded linear functional ϕ on P, i.e., $\phi \in P^*$, is attained at some point $p = p(\phi) \in P$.*

According to James' theorem [11] (see also [10, p. 643], [17, Ch. 1, Sect. 2, Corollary 2.4]), this property is equivalent to the reflexivity of the space P. Note that property (R) is fulfilled if the subspace P is finite-dimensional.

If a functional $F \in X^*$, $F \neq 0$, attains its norm at an element $x \in X$, $x \neq 0$, and if $F(x) > 0$, or—which is the same in this case—if

$$F(x) = \|F\|_{X^*}\, \|x\|_X, \tag{1.8}$$

we will say that the functional F possesses the N-property at the element x, or, shortly, the $N[x]$-property. By the complex variant of the Hahn–Banach Theorem (cf. [8, Ch. II, Sect. 3, Theorem 11]

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A CHARACTERIZATION OF EXTREMAL ELEMENTS IN SOME LINEAR PROBLEMS[1]

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Abstract: We give a characterization of elements of a subspace of a complex Banach space with the property that the norm of a bounded linear functional on the subspace is attained at those elements. In particular, we discuss properties of polynomials that are extremal in sharp pointwise Nikol'skii inequalities for algebraic polynomials in a weighted L_q-space on a finite or infinite interval.

Key words: Complex Banach space, Bounded linear functional on a subspace, Algebraic polynomial, Pointwise Nikol'skii inequality.

1. Bounded linear functionals in complex Banach spaces

1.1. Introduction. Statement of the problem

Let $X = X_{\mathbb{C}}$ be a complex Banach space (more precisely, a Banach space over the field $\mathbb{C}$ of complex numbers), let $S(X)$ be its unit sphere, and let $X^* = X^*_{\mathbb{C}}$ be the dual space of X, i.e., the space of complex-valued bounded linear (over the field $\mathbb{C}$ of complex numbers) functionals F on X with the norm

$$\|F\|_{X^*} = \sup\{|F(x)|\colon x \in X,\ \|x\|_X = 1\}.$$

Let P be a (closed) subspace of X, and let ψ be a bounded linear functional on P. We denote by

$$D(\psi; P) = \sup\{|\psi(p)|\colon p \in P,\ \|p\|_X = 1\} \tag{1.1}$$

the norm of the functional ψ on the subspace P. In what follows, we assume that $\psi \not\equiv 0$, so that $D(\psi; P) > 0$. The value $D(\psi; P)$ is the smallest possible (the best) constant in the inequality

$$|\psi(p)| \le D(\psi; P)\|p\|_X, \quad p \in P. \tag{1.2}$$

Nonzero elements p of the subspace P with the property that inequality (1.2) turns into an equality for them (if such elements exist) will be called extremal elements in this inequality. Elements p of the unit sphere $S(P) = S(X) \cap P$ of the subspace P that solve problem (1.1), i.e., those with the property that the supremum in (1.1) is attained at p, will be called extremal elements in problem (1.1). We will use the same terminology also in other similar situations. It is clear that an element $\varrho \in P$ is extremal in inequality (1.2) if and only if the element $\varrho/\|\varrho\|_X$ is extremal in problem (1.1). In this sense, extremal elements in problem (1.1) and inequality (1.2) coincide. The aim of this papers is exactly to characterize extremal elements in inequality (1.2) or in problem (1.1), which is the same.

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REFERENCES

1. **Luzin N.N.** Integral and trigonometric series. Moscow – Leningrad: GITTL. 1951. 550 p. [in Russian]
2. **Kolmogoroff A.** Une série de Fourier – Lebesgue divergente preque partout // Fund. math., 1923. Vol. 4. P. 324–328.
3. **Kolmogoroff A.** Une série de Fourier – Lebesgue divergente partout // C. r. Acad. sci. Paris, 1926. Vol. 183. P. 1327–1329.
4. **Carleson L.** On convergence and growth of partial sums of Fourier series // Acta math., 1966. Vol. 116, no. 1–2. P. 135–157. DOI: 10.1007/BF02392815
5. **Hunt R.A.** On the convergence of Fourier series // Orthogonal expansions and their continuous analogues. SIU Press, Carbondale, Illinois, 1968. P. 235–255.
6. **Sjölin P.** An inequality of Paley and convergence a.e. of Walsh-Fourier series // Arkiv för mat., 1969. Vol. 7. P. 551–570. DOI: 10.1007/BF02590894
7. **Antonov N.Yu.** Convergence of Fourier series // East Journal on Approximations, 1996. Vol. 2, no. 2. P. 187–196.
8. **Konyagin S.V.** On everywhere divergence of trigonometric Fourier series// Sb. Math., 2000. Vol. 191, no. 1. P. 97–120. DOI: 10.1070/SM2000v191n01ABEH000449
9. **Tevzadze N.R.** The convergence of the double Fourier series of a square integrable function // Soobshch. AN GSSR, 1970. Vol. 58, no. 2. P. 277–279. [in Russian]
10. **Fefferman C.** On the convergence of multiple Fourier series // Bull. Amer. Math. Soc., 1971. Vol. 77, no. 5. P. 744–745. DOI: 10.1090/S0002-9904-1971-12793-3
11. **Sjölin P.** Convergence almost everywhere of ertain singular integrals and multiple Fourier series // Arkiv för mat., 1971. Vol. 9, no. 1. P. 65–90. DOI: 10.1007/BF02383638
12. **Antonov N.Yu.** Almost everywhere convergence over cubes of multiple trigonometric Fourier series // Izv. Math., 2004. Vol. 68, no. 2, P. 223–241. DOI: 10.1070/IM2004v068n02ABEH000472
13. **Antonov N.Yu.** On the almost everywhere convergence of sequences of multiple rectangular Fourier sums // Proc. Steklov Inst. Math., 2009. Vol. 264, Suppl. 1. P. S1–S18. DOI: 10.1134/S0081543809050010
14. **Konyagin S.V.** On divergence of trigonometric Fourier series over cubes // Acta Sci. Math. (Szeged), 1995. Vol. 61. P. 305–329.
15. **Fefferman C.** On the divergence of multiple Fourier series // Bull. Amer. Math. Soc., 1971. Vol. 77, no. 2. P. 191–195. DOI: 10.1090/S0002-9904-1971-12675-7
16. **Bakhbukh M., Nikishin E. M.** The convergence of the double Fourier series of continuous functions // Siberian Math. Journal, 1973. Vol. 14, no. 6. P. 832–839. DOI: 10.1007/BF00975888
17. **Bakhvalov A. N.** Divergence everywhere of the Fourier series of continuous functions of several variables // Sb. Math., 1997. Vol. 188, no. 8. P. 1153–1170. DOI: 10.1070/SM1997v188n08ABEH000240.
18. **Bakhvalov A. N.** λ-divergence of the Fourier series of continuous functions of several variables // Mathematical Notes, 2002. Vol. 72, no. 3–4. P. 454–465. DOI: 10.4213/mzm438
19. **Antonov N.Yu.** On divergence almost everywhere of Fourier series of continuous functions of two variables // Izvestiya of Saratov University. New ser. Ser. Math. Mech. Inform., 2014. Vol. 14, iss. 4, pt. 2. P. 495–502. [in Russian]
20. **Stein E.M.** On limits of sequences of operators // Ann. Math., 1961. Vol. 74, no. 1. P. 140–170.
21. **Zygmund A.** Trigonometric series, vol. 1. New York: Cambridge Univ. Press, 1959. 383 p.

Further, let $f \in \varphi_d(L)(\mathbb{T}^d)$ and $y > 0$. Suppose that

$$g(x) = g_y(x) = \begin{cases} f(x), & |f(x)| > y, \\ 0, & |f(x)| \le y; \end{cases} \qquad h(x) = h_y(x) = f(x) - g(x).$$

Define $\lambda_f(y) = \operatorname{mes}\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y\}$. Then

$$\lambda_f(y) \le \operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(g, \mathbf{x}) > y/2\right\} + \operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(h, \mathbf{x}) > y/2\right\} = \lambda_g(y/2) + \lambda_h(y/2).$$

From this, using the equality

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} = -\int\limits_0^\infty y\, d\lambda_f(y) = \int\limits_0^\infty \lambda_f(y)\, dy$$

(see, for example, [21, Chapter 1, § 13, formula (13.6)]), we obtain

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le \bar{y}_d(2\pi)^d + \int\limits_{\bar{y}_d}^\infty \lambda_f(y)\, dy \le \bar{y}_d(2\pi)^d + \int\limits_{\bar{y}_d}^\infty \lambda_g\left(\frac{y}{2}\right) dy + \int\limits_{\bar{y}_d}^\infty \lambda_h\left(\frac{y}{2}\right) dy. \tag{16}$$

Taking into account that $g \in \varphi_d(L)(\mathbb{T}^d)$ and $h \in L^\infty(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$ and applying estimate (15) to $\lambda_g(y/2)$ and estimate (14) to $\lambda_h(y/2)$, from (16), we obtain

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le \bar{y}_d(2\pi)^d + 2\bar{A}_d \int\limits_{\bar{y}_d}^\infty \left(\frac{1}{y}\int\limits_{\mathbb{T}^d} \varphi_d(|g(\mathbf{t})|)\, d\mathbf{t}\right) dy + 4A_d^2 \int\limits_{\bar{y}_d}^\infty \left(\frac{1}{y^2}\int\limits_{\mathbb{T}^d} |h(\mathbf{t})|^2\, d\mathbf{t}\right) dy =$$

$$= \bar{y}_d(2\pi)^d + 2\bar{A}_d \int\limits_{\bar{y}_d}^\infty \left(\frac{1}{y}\int\limits_{\{\mathbf{t}\in\mathbb{T}^d:\ |f(\mathbf{t})|>y\}} \varphi_d(|f(\mathbf{t})|)\, d\mathbf{t}\right) dy + 4A_d^2 \int\limits_{\bar{y}_d}^\infty \left(\frac{1}{y^2}\int\limits_{\{\mathbf{t}\in\mathbb{T}^d:\ |f(\mathbf{t})|\le y\}} |f(\mathbf{t})|^2\, d\mathbf{t}\right) dy. \tag{17}$$

Applying Fubibi's theorem to the integrals on the right hand side of (17), we conclude that

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le 2\bar{A}_d \int\limits_{\{\mathbf{t}\in\mathbb{T}^d:\ |f(\mathbf{t})|>\bar{y}_d\}} \varphi_d(|f(\mathbf{t})|)\left(\int\limits_{\bar{y}_d}^{|f(\mathbf{t})|} \frac{dy}{y}\right) d\mathbf{t} +$$

$$+ 4A_d^2 \int\limits_{\mathbb{T}^d} |f(\mathbf{t})|^2 \left(\int\limits_{|f(\mathbf{t})|}^\infty \frac{dy}{y^2}\right) d\mathbf{t} + \bar{y}_d(2\pi)^d,$$

hence, statement (6) follows easily.

Finally, the Λ-convergence of the Fourier series of an arbitrary function from the class $\varphi_d(L)(\mathbb{T}^d)$ can be obtained from (5) by means of standard arguments (see, for example, [12, Lemma 3]). Theorem 1 is proved. $\square$

$$= S_{\mathbf{n}'_k}\left(g_{k,t^k}, (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d)\right)$$

and

$$M_k(f, \mathbf{x}) = \int\limits_{\mathbb{T}} \sup_{\mathbf{n}'_k \in \Omega'_\Lambda} \left| S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d)\right) \right| dx^k.$$

Further,

$$\operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : M_k(f, \mathbf{x}) > y\right\} = 2\pi \operatorname{mes}\left\{(x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d) \in \mathbb{T}^{d-1} : M_k(f, \mathbf{x}) > y\right\} \le$$

$$\le \frac{2\pi}{y} \int\limits_{\mathbb{T}^{d-1}} M_k(f, \mathbf{x})\, dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^d =$$

$$= \frac{2\pi}{y} \int\limits_{\mathbb{T}^d} \sup_{\mathbf{n}'_k \in \Omega'_\Lambda} \left| S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d)\right) \right| d\mathbf{x} =$$

$$= \frac{2\pi}{y} \int\limits_{\mathbb{T}} \left(\int\limits_{\mathbb{T}^{d-1}} \sup_{\mathbf{n}'_k \in \Omega'_\Lambda} \left| S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d)\right) \right| dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^d \right) dx^k. \tag{11}$$

From this, applying the induction hypothesis (more precisely, statement (6) for the dimension $d-1$) to the inner integral on the right hand part of (11), we obtain

$$\operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : M_k(f, \mathbf{x}) > y\right\} \le \frac{2\pi}{y} \int\limits_{\mathbb{T}} \left(B_{d-1} \int\limits_{\mathbb{T}^{d-1}} \varphi_d(|f(\mathbf{x})|)\, dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^d + 1 \right) dx^k \le$$

$$\le \frac{(2\pi)^2 B_{d-1}}{y} \left(\int\limits_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|)\, d\mathbf{x} + 1 \right). \tag{12}$$

According to (10),

$$\left\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y\right\} \subset \left\{\mathbf{x} \in \mathbb{T}^d : M(f, \mathbf{x}) > \frac{y}{2}\right\} \bigcup \left(\bigcup_{k=2}^{d} \left\{\mathbf{x} \in \mathbb{T}^d : M_k(f, \mathbf{x}) > \frac{\pi y}{2(d-1)C}\right\} \right). \tag{13}$$

Combining (13), (4) and (12), we obtain (5) with the constant $A_d = 2K_d + 8\pi(d-1)^2 B_{d-1} C$.

Now, we only need to prove the validity of statement (6). To this end, let us use statement (5) proved above.

From (5), it follows that the majorant $M_\Lambda(f, \mathbf{x})$ is finite almost everywhere on $\mathbb{T}^d$ for all $f \in \varphi_d(L)(\mathbb{T}^d)$, in particular, for all $f \in L^2(T^d)$. Applying Stein's theorem on limits of sequences of operators [20, Theorem 1], we see that the operator $M_\Lambda(f, \cdot)$ is of weak type $(2,2)$, i.e., there is a constant $A_d^2 > 0$ such that, for all $y > 0$ and $f \in L^2(T^d)$,

$$\operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y\right\} \le \frac{A_d^2}{y^2} \int\limits_{\mathbb{T}^d} |f(\mathbf{x})|^2\, d\mathbf{x}. \tag{14}$$

Similarly, from [20, Theorem 3], we can obtain the following refinement of statement (5): there is a constant $\bar{A}_d > 0$ such that, for all $y \ge \bar{y}_d/2 = \bar{A}_d$ and $f \in \varphi_d(L)(\mathbb{T}^d)$,

$$\operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y\right\} \le \int\limits_{\mathbb{T}^d} \varphi_d\left(\frac{\bar{A}_d|f(\mathbf{x})|}{y}\right) d\mathbf{x} \le \frac{\bar{A}_d}{y} \int\limits_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|)\, d\mathbf{x}. \tag{15}$$